Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

Journal of Functional Analysis 205 (2003) 271–305

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

The stable manifold theorem for non-linear stochastic systems with memory. I. Existence of the semiflow

Salah-Eldin A. Mohammed[a,*,1] and Michael K.R. Scheutzow[b,2]

[a] *Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, IL 62901, USA*
[b] *Fakultät II, Institut für Mathematik, MA 7-5, Technical University of Berlin, Strasse des 17 Juni 136, D-10623 Berlin, Germany*

Received 10 January 2000; revised 1 April 2002; accepted 24 June 2003

Communicated by L. Gross

Abstract

We consider non-linear stochastic functional differential equations (sfde's) on Euclidean space. We give sufficient conditions for the sfde to admit locally compact smooth cocycles on the underlying infinite-dimensional state space. Our construction is based on the theory of finite-dimensional stochastic flows and a non-linear variational technique. In Part II of this article, the above result will be used to prove a stable manifold theorem for non-linear sfde's.

MSC: primary 60H10; 60H20; secondary 60H25

Keywords: Stochastic semiflow; Stochastic functional differential equation (sfde)

1. Introduction: basic setting

This work consists of two parts I and II. The main objective is to prove a stable manifold theorem for stochastic differential systems with finite memory or stochastic

*Corresponding author.
E-mail addresses: salah@sfde.math.siu.edu (S.-E.A. Mohammed), ms@math.tu-berlin.de (M.K.R. Scheutzow).
URL: http://sfde.math.siu.edu.

[1] The research of this author is supported in part by NSF Grants DMS-9503702, DMS-9703852, DMS-9975462, DMS-0203368 and by MSRI, Berkeley, CA.
[2] The research of this author is supported in part by MSRI, Berkeley, CA.

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2002.04.001

functional differential equations (sfde's). Part I of this article is devoted to the construction of infinite-dimensional stochastic semiflows generated by a large class of regular sfde's. In Part II, we use the stochastic semiflow constructed in Part I in order to prove the existence of smooth local stable and unstable manifolds in the neighborhood of a hyperbolic stationary solution of the sfde. These manifolds are asymptotically invariant under the stochastic semiflow and intersect transversally at the stationary random point. Furthermore, the unstable manifold is finite-dimensional with a fixed non-random dimension. Like its deterministic counterpart, the existence of local stable/unstable manifolds is a central result in the theory of non-linear infinite-dimensional stochastic flows. The authors are not aware of any results to date on the existence of stable/unstable manifolds for non-linear infinite-dimensional stochastic differential systems.

In this section, we describe more precisely the general framework and hypotheses used in the subsequent analysis.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $\bar{\mathcal{F}}$ the P-completion of $\mathcal{F}$, and let $(\Omega, \bar{\mathcal{F}}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ be a complete filtered probability space satisfying the usual conditions [Pr.2]. Fix an arbitrary $r>0$ and a positive integer d.

Consider the stochastic functional differential equation (sfde):

$$\left.\begin{aligned} dx(t) &= H(t, x(t), x_t)\,\mathrm{d}t + G(t, x(t))\,\mathrm{d}W(t), \quad t\geqslant t_0\geqslant 0, \\ x(t_0) &= v\in\mathbf{R}^d, \quad x_{t_0} = \eta\in L^2([-r,0],\mathbf{R}^d). \end{aligned}\right\} \tag{I}$$

A solution of (I) is a process $x : [t_0 - r, \infty) \times \Omega \to \mathbf{R}^d$ whereby x_t denotes the segment

$$x_t(\cdot,\omega)(s) := x(t+s,\omega), \quad s\in[-r,0],\ \omega\in\Omega,\ t\geqslant t_0,$$

and (I) holds a.s. As state space, we will use the Hilbert space $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$ furnished with the norm

$$||(v,\eta)||_{M_2} := (|v|^2 + ||\eta||_{L^2}^2)^{1/2}, \quad v\in\mathbf{R}^d,\ \eta\in L^2([-r,0],\mathbf{R}^d).$$

The drift is a non-linear functional $H:\ \mathbf{R}^+ \times M_2 \to \mathbf{R}^d$, the noise coefficient is a mapping $G:\ \mathbf{R}^+ \times \mathbf{R}^d \to \mathbf{R}^{d\times p}$, and W is p-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$.

Denote $\Delta := \{(s,t)\in\mathbf{R}^2 : 0\leqslant s\leqslant t\}$. For a topological space E, let $\mathcal{B}(E)$ denote the Borel σ-algebra of E.

Let k be a positive integer and $0<\delta\leqslant 1$. If E, N are real Banach spaces, we will denote by $L^k(E,N)$ the Banach space of all continuous k-multilinear maps $A : E^k \to N$ with the uniform norm $||A|| := \sup\{|A(v_1, v_2, \dots, v_k)| : v_i\in E, |v_i|\leqslant 1, i = 1,\dots,k\}$. Suppose $U\subseteq E$ is an open set. A map $f : U\to N$ is said to be of class $C^{k,\delta}$ if it is C^k and if $D^k f : U \to L^k(E,N)$ is δ-Hölder continuous on bounded sets in U. A $C^{k,\delta}$ map $f : U\to N$ is said to be of class $C_b^{k,\delta}$ if all its derivatives $D^j f, 1\leqslant j\leqslant k$, are globally bounded on U, and $D^k f : U\to L^k(E,N)$ is δ-Hölder continuous on U. A mapping $\tilde{f}$: $[0,T]\times U\to N$ is *of class $C^{k,\delta}$ in the second variable uniformly with respect* *to the first* if for each $t \in [0,T]$, $\tilde{f}(t,\cdot)$ is $C^{k,\delta}$ on every bounded set in U and the corresponding δ-Hölder constant of $D_2^k \tilde{f}(t,\cdot)$ is uniformly bounded in $t \in [0,T]$. A mapping $\tilde{f}: [0,T] \times U \to N$ is *of class $C_b^{k,\delta}$ in the second variable uniformly with respect to the first*, if for each $t \in [0,T]$, $\tilde{f}(t,\cdot)$ is $C_b^{k,\delta}$ on U, the spatial derivatives $D_2^j \tilde{f}(t,x), 1 \leqslant j \leqslant k$, are globally bounded in $(t,x) \in [0,T] \times U$, and the corresponding δ-Hölder constant of $D_2^k \tilde{f}(t,\cdot)$ (over the whole of U) is uniformly bounded in $t \in [0,T]$.

The following definitions are crucial to the developments in this article.

Definition 1.1. Let E be a Banach space, k a non-negative integer and $\varepsilon \in (0,1]$. A *stochastic $C^{k,\varepsilon}$ semiflow* on E is a random field $X : \Delta \times E \times \Omega \to E$ satisfying the following properties:

(i) X is $(\mathscr{B}(\Delta) \otimes \mathscr{B}(E) \otimes \mathscr{F}, \mathscr{B}(E))$-measurable.
(ii) For each $\omega \in \Omega$, the map $\Delta \times E \ni (s,t,x) \mapsto X(s,t,x,\omega) \in E$ is continuous.
(iii) For fixed $(s,t,\omega) \in \Delta \times \Omega$, the map $E \ni x \mapsto X(s,t,x,\omega) \in E$ is $C^{k,\varepsilon}$.
(iv) If $0 \leqslant s \leqslant t \leqslant u$, $\omega \in \Omega$ and $x \in E$, then

$$X(s,u,x,\omega) = X(t,u,X(s,t,x,\omega),\omega).$$

(v) For all $(s,x,\omega) \in \mathbf{R}^+ \times E \times \Omega$, one has $X(s,s,x,\omega) = x$.

Definition 1.2. Let $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ be a P-preserving semigroup on the probability space $(\Omega, \mathscr{F}, P)$, E a Banach space, k a non-negative integer and $\varepsilon \in (0,1]$. A $C^{k,\varepsilon}$ *perfect cocycle* (Y,θ) on E is a $(\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{B}(E) \otimes \mathscr{F}, \mathscr{B}(E))$-measurable random field $Y : \mathbf{R}^+ \times E \times \Omega \to E$ with the following properties:

(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t,x) \mapsto Y(t,x,\omega) \in E$ is continuous; and for fixed $(t,\omega) \in \mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto Y(t,x,\omega) \in E$ is $C^{k,\varepsilon}$.
(ii) $Y(t+s,\cdot,\omega) = Y(t,\cdot,\theta(s,\omega)) \circ Y(s,\cdot,\omega)$ for all $s,t \in \mathbf{R}^+$ and all $\omega \in \Omega$.
(iii) $Y(0,x,\omega) = x$ for all $x \in E, \omega \in \Omega$.

We shall frequently refer to (ii) in Definition 1.2 as *the perfect cocycle identity*.

The main objective of Part I of this article is to show that under sufficient regularity conditions on the coefficients, the sfde (I) admits a stochastic semiflow $X : \Delta \times M_2 \times \Omega \to M_2$ satisfying $X(t_0,t,(v,\eta),\cdot) = (x(t),x_t)$ for all $(v,\eta) \in M_2$ and $t \geqslant t_0$, a.s., where x is the solution of (I). We first construct a Fréchet differentiable stochastic local semiflow for (I) using a non-linear variational approach, based on the theory of finite-dimensional stochastic flows of diffeomorphisms on Euclidean space (Theorem 2.1, Section 2)(cf. Bismut [Bi]). A global locally compact semiflow $X : \Delta \times M_2 \times \Omega \to M_2$ for (I) is established in Section 3 (Theorem 3.1) under suitable growth conditions on H and G. Furthermore, if H and G are autonomous (or even stationary) and W is a helix with respect a P-preserving shift θ on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, P)$, then the semiflow X generates a $C^{k,\varepsilon}$ cocycle $(\hat{X},\theta)$ in the sense of Definition 1.2.

Theorem 4.1 forms a basis for a proof of the *Stable Manifold Theorem* for the sfde (I) (Part II, Theorem 3.1). The latter theorem is the central result of this article. For a precise statement and details of its proof see Part II.

Linear versions of the sfde (I) are studied in [Mo.2,Mo.3,Mo.4,M-S.1,M-S.2].

During the last two decades, the theory of stochastic flows on Euclidean space or finite-dimensional manifolds has received a great deal of attention from stochastic analysts and geometers. Such finite-dimensional stochastic flows arise naturally from stochastic differential systems without memory (viz. the case $r = 0$ in (I)). The reader may consult works by Baxendale [Ba], Bismut [Bi], Elworthy [El], Kunita [Ku], [M-S.3], [I-S], [M-S.4] and others. On the other hand, there is little work on non-linear stochastic flows in infinite dimensions. Infinite-dimensional linear stochastic semiflows arise naturally from so-called *regular* stochastic fde's and certain classes of stochastic partial differential equations (spde's). See [Mo.2,Mo.3,Mo.4, M-S.1,M-S.2,Fl.1,Fl.2] and [F-S]. One should note that the question of existence of infinite-dimensional stochastic flows is highly non-trivial. In particular, the *singular* one-dimensional linear stochastic delay equation

$$dx(t) = x(t-1)\,\mathrm{d}W(t), \quad t \geqslant 0.$$

does not admit a continuous (or even linear) stochastic semiflow on M_2 ([Mo.4], [Mo.1], pp. 144–149, [M-S.2]).

In Section 5, we will show that the methods in this article cover a much larger class of sfde's than (I). In particular, one can handle equations of the form:

$$dx(t) = H(t, x(t-d_m), \ldots, x(t-d_1), x(t), x_t)\mu(dt) + G(dt, x(t), g(x_t)), \quad t \geqslant t_0 \geqslant 0,$$

$$x(t_0) = v \in \mathbf{R}^d, \quad x_{t_0} = \eta \in L^2([-r, 0], \mathbf{R}^d).$$

In the above equation, $G: \mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^m \times \Omega \to \mathbf{R}^d$ is a Kunita-type spatial local martingale [Ku], μ is a process of locally bounded variation; and for each $x \in L^2([-r, T], \mathbf{R}^d)$, the map $\mathbf{R} \ni t \mapsto g(x_t) \in \mathbf{R}^m$ is locally of bounded variation. For further details, see Section 5. However, for simplicity of exposition, our analysis will focus on (I).

In order to specify the regularity conditions on H and G in (I), we introduce the following notation and hypotheses:

Hypotheses $(\mathbf{W_{k,\delta}})$

(1) $H: \mathbf{R}^+ \times M_2 \to \mathbf{R}^d$ is jointly continuous; for each $T > 0$ and $t \in [0, T]$, the map $M_2 \ni (v, \eta) \mapsto H(t, v, \eta) \in \mathbf{R}^d$ is Lipschitz on bounded sets in M_2 uniformly with respect to $t \in [0, T]$; for each $T > 0$ and $t \in [0, T]$, the map $M_2 \ni (v, \eta) \mapsto H(t, v, \eta) \in \mathbf{R}^d$ is $C^{k,\delta}$ uniformly with respect to $t \in [0, T]$.

(2) $G: \mathbf{R}^+ \times \mathbf{R}^d \to \mathbf{R}^{d \times p}$ is jointly continuous; for each $T > 0$, $G|\{[0, T] \times \mathbf{R}^d\}$ is of class $C_b^{k+1,\delta}$ in the second variable uniformly with respect to $t \in [0, T]$.

In the next section, we will establish the existence of a smooth local semiflow for (I) under the above hypotheses.

2. Construction of the local semiflow

Our objective in this section is to show that the sfde (I) admits trajectories $\{(x(t), x_t) \in M_2 : t \geqslant t_0,\ (x(t_0), x_{t_0}) = (v, \eta) \in M_2\}$ which generate a continuous local semiflow on M_2.

The two key steps in the construction of the local semiflow are as follows:

(i) Using the stochastic flow of diffeomorphisms generated by the noise coefficient G, we construct a random integral equation which is equivalent to the sfde (I).

(ii) By a fixed-point argument we show that the random integral equation has a smooth local semiflow which (by (i)) agrees with the local trajectories of the sfde (I) prior to explosion times.

In this section and the rest of the article, we will adopt the following notation:

If $\rho \in \mathbf{R}^+$ and $(v, \eta) \in M_2$, denote by $B((v, \eta), \rho)$ the open ball in M_2 with center (v, η) and radius ρ. The corresponding closed ball is denoted by $\bar{B}((v, \eta), \rho)$.

$$\widetilde{(v, \eta)}(t) := \begin{cases} v, & t \geqslant 0, \\ \eta(t), & -r \leqslant t < 0, \end{cases} \quad (v, \eta) \in M_2.$$

For $t_0 \geqslant 0$ and a stopping time $T \geqslant t_0$, define $\tilde{\mathscr{E}}_{t_0}^T := \{x : \{(t, \omega) : \omega \in \Omega,\ t_0 - r \leqslant t \leqslant T(\omega),\ t < \infty\} \to \mathbf{R}^d,\ x|[t_0 - r, t_0] \times \Omega$ is $(\mathscr{B}([t_0 - r, t_0]) \otimes \mathscr{F}_{t_0})$-measurable, $\int_{t_0-r}^{t_0} |x(s, \omega)|^2 \, \mathrm{d}s < \infty$ for all $\omega \in \Omega$, and x is continuous and adapted for $t \geqslant t_0\}$.

We will often delete the upper index T when $T \equiv \infty$ and write $\tilde{\mathscr{E}}_{t_0} := \tilde{\mathscr{E}}_{t_0}^{\infty}$.

For $0 \leqslant t_0 \leqslant T < \infty$ (non-random) and $M > 0$, define the following spaces:

$\mathscr{E}_{t_0}^T := \{x \in L^2([t_0 - r, T], \mathbf{R}^d) : x|[t_0, T]$ is continuous$\}$.

$\mathscr{E}_{t_0}^T(M) := \{x \in \mathscr{E}_{t_0}^T : \int_{t_0-r}^{t_0} |x(s)|^2 \, \mathrm{d}s + |x(t_0)|^2 + \sup_{t_0 \leqslant t \leqslant T} |x(t)|^2 \leqslant M^2\}$.

${}_0\mathscr{E}^T(M) := \{x \in \mathscr{E}_0^T(M) : (x(0), x_0) = 0\}$.

${}_0\mathscr{E}^T := \{x \in \mathscr{E}_0^T : (x(0), x_0) = 0\}$.

The spaces $\mathscr{E}_{t_0}^T$ and ${}_0\mathscr{E}^T$ are given the complete norms:

$$||x|| := \left\{ \int_{t_0-r}^{t_0} |x(s)|^2 \, \mathrm{d}s + |x(t_0)|^2 + \sup_{t_0 \leqslant t \leqslant T} |x(t)|^2 \right\}^{1/2}, \quad x \in \mathscr{E}_{t_0}^T,$$

and

$$||x||_\infty := \sup_{t_0 \leqslant t \leqslant T} |x(t)|, \quad x \in {}_0\mathscr{E}^T.$$

$\triangle := \{(x, x) : x \in \mathbf{R}^d\} \subset \mathbf{R}^d \times \mathbf{R}^d,\ \triangle^c := \mathbf{R}^d \times \mathbf{R}^d \backslash \triangle.$

$\alpha := (\alpha_1, \alpha_2, \dots, \alpha_d)$, $D^\alpha_x := \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_d)^{\alpha_d}}$, $|\alpha| := \sum_{i=1}^d \alpha_i$, for α_i non-negative integers, $i = 1, \dots, d$.

$C^{k,\delta}(\mathbf{R}^d, \mathbf{R}^d) := \{f : f : \mathbf{R}^d \to \mathbf{R}^d$ is $C^{k,\delta}\}$ is the Fréchet space furnished with the family of seminorms $\|\cdot\|_{k+\delta:K}$:

$$\|f\|_{k+\delta:K} := \sup_{x\in K} \frac{|f(x)|}{(1+|x|)} + \sum_{1\leqslant|\alpha|\leqslant k} \sup_{x\in K} |D^\alpha_x f(x)| + \sum_{|\alpha|=k} \sup_{(x,y)\in K\cap\triangle^c} \frac{|D^\alpha_x f(x) - D^\alpha_y f(y)|}{|x-y|^\delta}$$

for compact sets K in $\mathbf{R}^d$.

The Fréchet space $C^k(\mathbf{R}^d, \mathbf{R}^d)$ is similarly defined (where the seminorms $\|\cdot\|_{k:K}$ are defined without the last sum in the above expression).

In view of Hypotheses $(W_{k,\delta})(2)$, it follows from Kunita ([Ku], Theorem 4.6.5, p. 173) that the solutions of the sde

$$\left.\begin{array}{l} d\psi(t) = G(t, \psi(t))\, \mathrm{d}W(t), \quad t \geqslant 0, \\ \psi(0) = x \in \mathbf{R}^d \end{array}\right\} \tag{II}$$

generate a $(\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{B}(\mathbf{R}^d) \otimes \mathscr{F}, \mathscr{B}(\mathbf{R}^d))$-measurable random field $\psi : \mathbf{R}^+ \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ such that for each $x \in \mathbf{R}^d$, $\psi(0, x, \omega) = x$, and the process $\psi(t, x, \cdot)$, $t \geqslant 0$, is an $(\mathscr{F}_t)_{t\geqslant 0}$-local martingale. Furthermore, for each $t \geqslant 0$, $\omega \in \Omega$ and $\varepsilon \in (0, \delta)$, the map $\psi(t, \cdot, \omega)$ belongs to $\mathrm{Diff}^{k+1,\varepsilon}(\mathbf{R}^d, \mathbf{R}^d)$, the group of all $C^{k+1,\varepsilon}$ diffeomorphisms $\mathbf{R}^d \to \mathbf{R}^d$. The diffeomorphism group $\mathrm{Diff}^{k+1,\varepsilon}(\mathbf{R}^d, \mathbf{R}^d)$ is given the subspace topology from $C^{k+1,\varepsilon}(\mathbf{R}^d, \mathbf{R}^d)$, and for each $\omega \in \Omega$, the map

$$[0, T] \ni t \mapsto \psi(t, \cdot, \omega) \in \mathrm{Diff}^{k+1,\varepsilon}(\mathbf{R}^d, \mathbf{R}^d)$$

is continuous ([Ku], Theorem 4.6.5). See also [I-W]. We denote the partial Fréchet derivative of ψ with respect to x by $D\psi(t, x, \omega) \in L(\mathbf{R}^d)$.

Further, we define $\zeta : [0, \infty) \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ by

$$\zeta(t, x, \omega) := \psi(t, \cdot, \omega)^{-1}(x)$$

for all $t \geqslant 0$, $x \in \mathbf{R}^d, \omega \in \Omega$.

The following lemmas will be needed in the construction of the local semiflow of (I). The proofs of Lemmas 2.1 and 2.2 are straightforward. We will omit them. It is interesting to note (as will be apparent from the subsequent discussion) that the existence of the stochastic semiflow in Theorem 3.1 may be deduced by stipulating the conclusions of Lemma 2.1 together with appropriate continuity and differentiability hypotheses on H and G.

Lemma 2.1. *Assume that H satisfies Hypotheses $(W_{k,\delta})(1)$ for some $k\geqslant 0, \delta\in(0,1)$. Then the following is true*:

(i) *For each $t_0\geqslant 0$ and $x\in\tilde{\mathscr{E}}_{t_0}$, the map $[t_0,\infty)\times\Omega\ni(t,\omega)\mapsto H(t,x(t,\omega), x_t(\cdot,\omega))\in\mathbf{R}^d$ is progressively measurable.*

(ii) *There exists $L:[0,\infty)^3\to[0,\infty)$ such that*

$$\int_{t_0}^{t_0+t} |H(u,x(u),x_u)-H(u,y(u),y_u)|\,\mathrm{d}u\leqslant L(t,T,M)\sup_{t_0\leqslant u\leqslant t_0+t}|x(u)-y(u)|,$$

whenever $0\leqslant t_0\leqslant t_0+t\leqslant T$, $M\geqslant 0$, $x,y\in\mathscr{E}^T_{t_0}(M)$ and $x_{t_0}=y_{t_0}$, $x(t_0)=y(t_0)$. Furthermore, L is continuous in its three variables, and is such that $L(0,T,M)=0$ for all $T,M\geqslant 0$.

(iii) *For any $T,M\in(0,\infty)$, we have*

$$\lim_{t\downarrow 0}\sup_{0\leqslant t_0\leqslant T}\sup_{x\in\mathscr{E}^T_{t_0}(M)}\int_{t_0}^{t_0+t}|H(u,x(u),x_u)|\,\mathrm{d}u=0.$$

Lemma 2.2. *Suppose that G satisfies Hypothesis $(W_{k,\delta})(2)$ for some $k\geqslant 0, \delta\in(0,1)$. Then the stochastic flow ψ of* (II) *has the following properties*: *For each $\omega\in\Omega, T>0$, and $\varepsilon\in(0,\delta)$, the map*

$$\mathbf{R}^d\ni x\mapsto\{D\psi(\cdot,x,\omega)\}^{-1}\in C([0,T],L(\mathbf{R}^d))$$

is $C^{k,\varepsilon}$; and the map

$$\mathbf{R}^d\ni x\mapsto\zeta(\cdot,x,\omega)\in C([0,T],\mathbf{R}^d)$$

is $C^{k+1,\varepsilon}$.

Our first step in the construction of the stochastic semi-flow of (I) is to show that (I) is equivalent to a random integral equation which can be solved for each $\omega\in\Omega$.

Define the function $F:[0,\infty)\times\mathbf{R}^d\times M_2\times\Omega\to\mathbf{R}^d$ by

$$F(t,z,v,\eta,\omega):=\{D\psi(t,z,\omega)\}^{-1}H(t,v,\eta)$$

for all $t\geqslant 0, z,v\in\mathbf{R}^d,\eta\in L^2([-r,0],\mathbf{R}^d),\omega\in\Omega$. Fix $t_0\geqslant 0$, a stopping time $T\geqslant t_0$, $(v,\eta)\in M_2$ and consider the equation

$$x(t,\omega)=\begin{cases}\psi(t,[\zeta(t_0,v,\omega)+\int_{t_0}^t F(u,\zeta(u,x(u,\omega),\omega),x(u,\omega),x_u(\cdot,\omega),\omega)\,\mathrm{d}u],\omega), \\ \qquad\qquad t\in[t_0,T(\omega)]\cap[t_0,\infty), \\ \eta(t-t_0), \qquad \text{a.e. } t\in[t_0-r,t_0).\end{cases}\tag{III}$$

Note that for every $x \in \tilde{\mathscr{E}}_{t_0}$ the map $u \mapsto \zeta(u, x(u,\omega), \omega)$ is continuous and adapted. By Lemma 2.1(i) and Lemma 2.2, the integrand in the right-hand side of (III) is progressively measurable and integrable. Furthermore, the indefinite integral in (III) is continuous in t, adapted and of locally bounded variation. Finally, the right-hand side of (III) is continuous and adapted.

Lemma 2.3. *Assume Hypotheses* $(W_{k,\delta})$ *for some* $k \geqslant 0$, $\delta \in (0,1]$. *Fix* $t_0 \geqslant 0$, $(v,\eta) \in M_2$ *and a stopping time* $T \geqslant t_0$. *A process* $x \in \tilde{\mathscr{E}}_{t_0}^T$ *solves* (I) *on the interval* $[t_0, T(\omega)] \cap [t_0, \infty)$ *with initial condition* $(x(t_0), x_{t_0}) = (v, \eta)$ *iff* x *solves* (III) *for a.a.* $\omega \in \Omega$.

Proof. Assume that $x \in \tilde{\mathscr{E}}_{t_0}^T$ solves (III) for a.a. $\omega \in \Omega$. Then $(x(t_0), x_{t_0}) = (v, \eta)$. In order to cover the case $k = 0$, we will require a straightforward modification of the generalized Itô's formula in Kunita ([Ku], Theorem 3.3.1, p. 92, and Theorem 3.3.3(i), p. 94f). Using this fact, (III) and the remark preceding this lemma, it follows that x is a continuous semimartingale and

$$
\begin{aligned}
dx(t) &= \psi\Big(dt, \Big[\zeta(t_0, v, \cdot) + \int_{t_0}^{t} F(u, \zeta(u, x(u), \cdot), x(u), x_u, \cdot)\, \mathrm{d}u\Big], \cdot\Big) \\
&\quad + D\psi\Big(t, \Big[\zeta(t_0, v, \cdot) + \int_{t_0}^{t} F(u, \zeta(u, x(u), \cdot), x(u), x_u, \cdot)\, \mathrm{d}u\Big], \cdot\Big) \\
&\quad \times [F(t, \zeta(t, x(t), \cdot), x(t), x_t, \cdot)]\, \mathrm{d}t \\
&= G\Big(t, \psi\Big(t, \Big[\zeta(t_0, v, \cdot) + \int_{t_0}^{t} F(u, \zeta(u, x(u), \cdot), x(u), x_u, \cdot)\, \mathrm{d}u\Big], \cdot\Big)\, \mathrm{d}W(t) \\
&\quad + H(t, x(t), x_t)\, \mathrm{d}t \\
&= G(t, x(t))\, \mathrm{d}W(t) + H(t, x(t), x_t)\, \mathrm{d}t.
\end{aligned}
$$

Therefore x solves (I).

Conversely, suppose that $x \in \tilde{\mathscr{E}}_{t_0}^T$ solves (I) and define

$$\xi(t,\omega) := \zeta(t_0, v, \omega) + \int_{t_0}^{t} F(u, \zeta(u, x(u), \omega), x(u), x_u, \omega)\, \mathrm{d}u.$$

By the remark preceding the lemma, ξ is adapted, continuous and of locally bounded variation. Therefore, the same computation as above shows that the process

$$\tilde{x}(t,\omega) := \begin{cases} \psi(t, \xi(t,\omega), \omega), & t \geqslant t_0, \\ \eta(t - t_0), & t_0 - r \leqslant t \leqslant t_0, \end{cases}$$

is a semimartingale with differential

$$d\tilde{x}(t) = G(t, \tilde{x}(t))\, \mathrm{d}W(t) + H(t, x(t), x_t)\, \mathrm{d}t.$$

The above sde (without memory) has a unique solution $\tilde{x}$ with initial condition $\tilde{x}(t_0) = v$, so $\tilde{x} \equiv x$ a.s. on $[t_0 - r, T] \cap [t_0 - r, \infty)$. This proves the lemma. $\square$

The following theorem establishes the existence of a unique local semiflow of equation (III).

Theorem 2.1. *Assume Hypotheses* $(W_{k,\delta})$ *for some* $k \geqslant 1$, $\delta \in (0, 1]$. *Let* ∂ *be an element not contained in* $\mathbf{R}^d$ *and denote by* $\mathbf{R}^d_\partial := \mathbf{R}^d \cup \{\partial\}$, *the one-point compactification of* $\mathbf{R}^d$. *Then there exist a* $(\mathscr{B}(\Delta) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(\mathbf{R}^d_\partial))$*-measurable random field*

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v, \eta), \omega) \mapsto x^{t_0,(v,\eta)}(t, \omega) \in \mathbf{R}^d_\partial$$

and a $(\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}((0, \infty]))$*-measurable random field*

$$[0, \infty) \times M_2 \times \Omega \ni (t_0, (v, \eta), \omega) \mapsto \tau(t_0, (v, \eta), \omega) \in (0, \infty]$$

such that the following is true:

(i) $\tau(t_0, (v, \eta), \omega) > t_0$.
(ii) $x^{t_0,(v,\eta)}(t, \omega) = \partial$ *iff* $\tau(t_0, (v, \eta), \omega) \leqslant t$.
(iii) *For each* $(v, \eta) \in M_2$, $t_0 \geqslant 0$, $\omega \in \Omega$, $x^{t_0,(v,\eta)}(\cdot, \omega)$ *is the unique solution of* (III) *on* $[t_0, \tau(t_0, (v, \eta), \omega))$.
(iv) *If* $\tau(t_0, (v, \eta), \omega) < \infty$, *then* $\limsup_{t \uparrow \tau(t_0,(v,\eta),\omega)} |x^{t_0,(v,\eta)}(t)| = \infty$.
(v) *For each* $\omega \in \Omega$, *the map*

$$[0, \infty) \times M_2 \ni (t_0, (v, \eta)) \mapsto \tau(t_0, (v, \eta), \omega) \in (0, \infty]$$

is lower semicontinuous.

(vi) *For any* $(t_0, t, (v, \eta), \omega) \in \Delta \times M_2 \times \Omega$ *with* $t < \tau(t_0, (v, \eta), \omega)$, *set*

$$X(t_0, t, (v, \eta), \omega) := (x^{t_0,(v,\eta)}(t, \omega), x_t^{t_0,(v,\eta)}(\cdot, \omega)).$$

Define the family of sets

$$D(\omega) := \{(t_0, t, (v, \eta)) \in \Delta \times M_2 : \tau(t_0, (v, \eta), \omega) > t\}$$

for $\omega \in \Omega$, *and*

$$D_{t_0,t}(\omega) := \{(v, \eta) \in M_2 : \tau(t_0, (v, \eta), \omega) > t\}$$

for $(t_0, t) \in \Delta$, $\omega \in \Omega$. *Then* $D(\omega)$, $D_{t_0,t}(\omega)$ *are open subsets of* $\Delta \times M_2$ *and* M_2, *respectively. Furthermore, for each* $\omega \in \Omega$, *the map*

$$D(\omega) \ni (t_0, t, (v, \eta)) \mapsto X(t_0, t, (v, \eta), \omega) \in M_2$$

is continuous; and for fixed $(t_0, t, \omega) \in \Delta \times \Omega$, *the map*

$$D_{t_0,t}(\omega) \ni (v, \eta) \mapsto X(t_0, t, (v, \eta), \omega) \in M_2$$

is $C^{k,\varepsilon}$ *for any* $\varepsilon \in (0, \delta)$.

(vii) *If* $0 \leqslant s \leqslant t \leqslant u$, $\omega \in \Omega$, $(v, \eta) \in M_2$ *and* $\tau(s, (v, \eta), \omega) > t$, *then*

$$\tau(t, X(s, t, (v, \eta), \omega), \omega) = \tau(s, (v, \eta), \omega);$$

and if $\tau(s, (v, \eta), \omega) > u$, *then*

$$X(s, u, (v, \eta), \omega) = X(t, u, X(s, t, (v, \eta), \omega), \omega).$$

(viii) *For each* $t_0 \in \mathbf{R}^+$, $(v, \eta) \in M_2$, $\tau(t_0, (v, \eta), \cdot)$ *is a stopping time.*

(ix) *For each* $(t_0, t) \in \Delta$, $(v, \eta) \in M_2$, *the map* $\omega \mapsto x^{t_0,(v,\eta)}(t, \omega)$ *is* $(\mathscr{F}_t, \mathscr{B}(\mathbf{R}^d_\partial))$-*measurable.*

The proof of Theorem 2.1 uses a fixed point argument. Fix and suppress $\omega \in \Omega$ until further notice. For $T \in (0, \infty)$, define the map $U_T : [0, \infty) \times M_2 \times {}_0\mathscr{E}^T \to {}_0\mathscr{E}^T$ by

$$U_T(t_0, (v, \eta), \bar{x})(t) := \begin{cases} \psi(t_0 + t, \zeta(t_0, v) + V(t_0, (v, \eta), \bar{x})(t)) - v, & 0 \leqslant t \leqslant T, \\ 0, & -r \leqslant t < 0, \end{cases}$$

where

$$V(t_0, (v, \eta), \bar{x})(t) := \int_0^t F(t_0 + u, \zeta(t_0 + u, \bar{x}(u) + v), \bar{x}(u) + v, \bar{x}_u + \widetilde{(v, \eta)}_u)\, \mathrm{d}u,$$

for all $(t_0, (v, \eta), \bar{x}) \in [0, \infty) \times M_2 \times {}_0\mathscr{E}^T$. Recall that $\widetilde{(v, \eta)} : [-r, \infty) \to \mathbf{R}^d$ is defined by

$$\widetilde{(v, \eta)}(t) := \begin{cases} v, & t \geqslant 0, \\ \eta(t), & -r < t < 0. \end{cases}$$

For fixed $(t_0, (v, \eta)) \in [0, \infty) \times M_2$ it is easy to check that $\bar{x} \in {}_0\mathscr{E}^T$ is a fixed point of $U_T(t_0, (v, \eta), \cdot)$ if and only if

$$x(t) := \begin{cases} v + \bar{x}(t - t_0), & t_0 \leqslant t \leqslant t_0 + T, \\ \eta(t - t_0), & t_0 - r \leqslant t < t_0 \end{cases}$$

solves equation (III) on $[t_0 - r, t_0 + T]$.

The fixed-point argument in the proof of Theorem 2.1 requires the following lemmas.

Lemma 2.4. *Assume* $(W_{k,\delta})$ *for some* $k \geqslant 0$ *and* $\delta \in (0,1]$. *Fix any* $\delta_1, \delta_2, \delta_3 > 0$. *Then there exists* $\tau_1 = \tau_1(\delta_1, \delta_2, \delta_3) > 0$ *such that*

$$U_{\tau_2}([0,\delta_1] \times \bar{B}(0,\delta_2) \times {}_0\mathscr{E}^{\tau_2}(\delta_3)) \subseteq {}_0\mathscr{E}^{\tau_2}(\delta_3)$$

for all $0 < \tau_2 \leqslant \tau_1$.

Proof. Let $t_0 \in [0,\delta_1], (v,\eta) \in \bar{B}(0,\delta_2), \bar{x} \in {}_0\mathscr{E}^{\tau}(\delta_3)), 0 \leqslant t \leqslant \tau$. Then

$$\begin{aligned}|U_\tau(t_0,(v,\eta),\bar{x})(t)| \leqslant{}& |\psi(t_0+t, \zeta(t_0,v) + V(t_0,(v,\eta),\bar{x})(t)) - \psi(t_0+t,\zeta(t_0,v))| \\ &+ |\psi(t_0+t,\zeta(t_0,v)) - \psi(t_0,\zeta(t_0,v))| \end{aligned} \tag{2.1}$$

where

$$V(t_0,(v,\eta),\bar{x})(t) := \int_0^t F(t_0+u, \zeta(t_0+u,\bar{x}(u)+v), \bar{x}(u)+v, \bar{x}_u + \widetilde{(v,\eta)}_u)\, \mathrm{d}u. \tag{2.2}$$

Observe that $[D\psi(\cdot,\cdot)]^{-1}$ and ζ take bounded sets to bounded sets (because they are continuous, Lemma 2.2). Using the definition of $F(t,z,v,\eta) := \{D\psi(t,z)\}^{-1}H(t,v,\eta)$, it follows from (2.2) and Lemma 2.1(iii) that

$$\lim_{\tau \downarrow 0} \sup_{t_0 \in [0,\delta_1]} \sup_{(v,\eta) \in \bar{B}(0,\delta_2)} \sup_{\bar{x} \in {}_0\mathscr{E}^{\tau}(\delta_3)} \sup_{0 \leqslant t \leqslant \tau} |V(t_0,(v,\eta),\bar{x})(t)| = 0. \tag{2.3}$$

Finally, use (2.1), (2.3) and the fact that ψ is jointly uniformly continuous on bounded sets, in order to conclude that

$$\lim_{\tau \downarrow 0} \sup_{t_0 \in [0,\delta_1]} \sup_{(v,\eta) \in \bar{B}(0,\delta_2)} \sup_{\bar{x} \in {}_0\mathscr{E}^{\tau}(\delta_3)} \sup_{0 \leqslant t \leqslant \tau} |U_\tau(t_0,(v,\eta),\bar{x})(t)| = 0.$$

The above relation implies the assertion of the lemma. □

The next lemma is elementary. For easy reference we give a proof in Appendix A.

Lemma 2.5. *Let* $k \geqslant 0$ *be an integer*, $T > 0$ *and* $\varepsilon \in [0,1)$. *Let* N *be a real Banach space*, $\Lambda \subseteq N$ *an open subset of* N, *and* μ *a function of bounded variation on* $[0,T]$. *Suppose the maps* $f : [0,T] \times \Lambda \to \mathbf{R}^d$ *and* $\psi : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$ *are jointly continuous*, *and* f *takes bounded sets to bounded sets. Define the maps* $I, U : \Lambda \to C([0,T],\mathbf{R}^d)$ *by*

$$\begin{aligned} I(\lambda)(t) &:= \int_0^t f(u,\lambda)\mu(du), \\ U(\lambda)(t) &:= \psi(t, f(t,\lambda)) \end{aligned}$$

for all $\lambda \in \Lambda$ *and* $t \in [0, T]$. *Then* I *and* U *are continuous* (*and take bounded sets to bounded sets*). *If further,* f *and* ψ *are* $C^{k,\varepsilon}$ *in the second variable uniformly with respect to* $t \in [0, T]$, *then* I *and* U *are* $C^{k,\varepsilon}$.

Lemma 2.6. *Assume* $(W_{k,\delta})$ *for some* $k \geqslant 1$, $\delta \in (0, 1]$. *Then the following is true*:

(i) *For any* $T > 0$, *the map* $U_T : [0, \infty) \times M_2 \times {}_0\mathscr{E}^T \to {}_0\mathscr{E}^T$ *is continuous. For fixed* $t_0 \geqslant 0$ *and any* $\varepsilon \in (0, \delta)$, *the map* $U_T(t_0, \cdot, \cdot) : M_2 \times {}_0\mathscr{E}^T \to {}_0\mathscr{E}^T$ *is* $C^{k,\varepsilon}$.

(ii) *Fix* $\delta_1, \delta_2, \delta_3 > 0$ *and let* $\tau_1 > 0$ *be as in Lemma* 2.4. *Then there exists* $\tau_2 \in (0, \tau_1]$ *such that*

$$\sup_{0 \leqslant t \leqslant \tau_2} |U_{\tau_2}(t_0, (v, \eta), x^1)(t) - U_{\tau_2}(t_0, (v, \eta), x^2)(t)| \leqslant \tfrac{1}{2} \sup_{0 \leqslant t \leqslant \tau_2} |x^1(t) - x^2(t)|$$

for all $(t_0, (v, \eta)) \in [0, \delta_1] \times \bar{B}(0, \delta_2)$ *and* $x^1, x^2 \in {}_0\mathscr{E}^{\tau_2}(\delta_3)$.

Proof.

(i) Recall the definition of $U_T : [0, \infty) \times M_2 \times {}_0\mathscr{E}^T \to {}_0\mathscr{E}^T$, viz.

$$U_T(t_0, (v, \eta), \bar{x})(t) := \begin{cases} \psi(t_0 + t, \zeta(t_0, v) + V(t_0, (v, \eta), \bar{x})(t)) - v, & 0 \leqslant t \leqslant T, \\ 0, & -r \leqslant t < 0, \end{cases}$$

where

$$V(t_0, (v, \eta), \bar{x})(t) := \int_0^t F(u + t_0, \zeta(u + t_0, \bar{x}(u) + v), \bar{x}(u) + v, \bar{x}_u + \widetilde{(v, \eta)}_u)\, \mathrm{d}u$$

for all $(t_0, (v, \eta), \bar{x}) \in [0, \infty) \times M_2 \times {}_0\mathscr{E}^T$. In the above expression, it is easy to see that the integrand is jointly continuous in $(t_0, u, (v, \eta), \bar{x})$. This follows from Hypothesis $(W_{k,\delta})(1)$, the continuity of the maps

$$[0, T] \times M_2 \ni (u, v, \eta) \mapsto \widetilde{(v, \eta)}_u \in M_2,$$

$$[0, \infty) \times [0, T] \ni (t_0, u) \mapsto H(u + t_0, \bar{x}(u) + v, \bar{x}_u + \widetilde{(v, \eta)}_u) \in \mathbf{R}^d$$

and Lemma 2.2. Hence by Lemma 2.5, this proves the continuity of $U_T(t_0, (v, \eta), \bar{x})$ in $(t_0, (v, \eta), \bar{x})$. Similarly, using Hypothesis $(W_{k,\delta})(1)$ and Lemmas 2.2, 2.5, the reader may show that the map $U_T(t_0, \cdot, \cdot) : M_2 \times {}_0\mathscr{E}^T \to {}_0\mathscr{E}^T$ is $C^{k,\varepsilon}$. This proves part (i) of the lemma.

(ii) Fix $\delta_1, \delta_2, \delta_3 > 0$ and let $\tau_1 > 0$ be as in Lemma 2.4. Suppose $(t_0, (v, \eta)) \in [0, \delta_1] \times \bar{B}(0, \delta_2), x^1, x^2 \in {}_0\mathscr{E}^{\tau_2}(\delta_3)$ and $0 \leqslant t \leqslant \tau_2 \leqslant \tau_1$. Since the maps $(t, v) \mapsto [D\psi(t, v)]^{-1}$, $(t, v) \mapsto \zeta(t, v)$ are continuous, an elementary computation shows that there is a positive constant $K = K(\delta_1, \delta_2, \delta_3)$ such that

$$
\begin{aligned}
&|U_{\tau_2}(t_0,(v,\eta),x^1)(t) - U_{\tau_2}(t_0,(v,\eta),x^2)(t)| \\
&\quad \leqslant K\Bigg(\int_0^t |H(t_0+u, x^1(u)+v, x_u^1 + \widetilde{(v,\eta)}_u) \\
&\qquad - H(t_0+u, x^2(u)+v, x_u^2 + \widetilde{(v,\eta)}_u)| \, \mathrm{d}u \\
&\qquad + \sup_{0 \leqslant u \leqslant t} ||[D\psi(t_0+u, x^1(u)+v)]^{-1} - [D\psi(t_0+u, x^2(u)+v)]^{-1}|| \\
&\qquad \times \int_0^t |H(t_0+u, x^1(u)+v, x_u^1 + \widetilde{(v,\eta)}_u)| \, \mathrm{d}u \Bigg)
\end{aligned}
$$

for all $0 < \tau_2 \leqslant \tau_1$. Assertion (ii) of the lemma now follows from the above inequality, Lemma 2.1(ii), (iii), and the fact that $[D\psi(t,v)]^{-1}$ has first partial derivatives with respect to v that are jointly continuous in (t,v) (Lemma 2.2). □

The following lemma is classical. See [H], Lemma 4.2, p. 46, and Appendix A.

Lemma 2.7. *Let E and N be real Banach spaces, $\Lambda \subseteq N$ an open subset and $B \subset E$ a closed ball. Let $\varepsilon \in [0,1)$. Suppose $U : \Lambda \times B \to B$ is a $C^{k,\varepsilon}$ map ($k \geqslant 0$) which is a contraction in the second variable uniformly with respect to the first, viz. there is a constant $L \in (0,1)$ such that*

$$|U(\lambda, x_1) - U(\lambda, x_2)| \leqslant L|x_1 - x_2|$$

for all $x_1, x_2 \in B$ and all $\lambda \in \Lambda$. Then for each $\lambda \in \Lambda$, there is a unique $x(\lambda) \in B$ such that $U(\lambda, x(\lambda)) = x(\lambda)$. Furthermore, the map $\Lambda \ni \lambda \mapsto x(\lambda) \in B \subset E$ is $C^{k,\varepsilon}$.

Note that, so far, the only place where we have required $k \geqslant 1$ is in the proof of Lemma 2.6(ii). In the above proof, we used the fact that $[D\psi(\cdot,\cdot)]^{-1}$ is locally Lipschitz in the spatial variable.

The following lemma is key to establishing the measurability properties of the semiflow in Theorem 2.1.

Lemma 2.8. *Assume $(W_{k,\delta})$ for some $k \geqslant 1$, $\delta \in (0,1]$, and suppose that H is globally bounded. Assume also that the neutral equation* (III) *admits at most one solution in $\mathscr{E}_{t_0}^T$ for each $\omega \in \Omega, (t_0,(v,\eta)) \in [0,\infty) \times M_2, T \geqslant t_0$. Then for each $\omega \in \Omega$, and any $(t_0,(v,\eta)) \in [0,\infty) \times M_2$,* (III) *has a* (*unique*) *solution $x^{t_0,(v,\eta)}(\cdot,\omega) : [t_0 - r, \infty) \to \mathbf{R}^d$ satisfying the following properties*:

(i) *The random field*

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v,\eta), \omega) \mapsto x^{t_0,(v,\eta)}(t,\omega) \in \mathbf{R}^d$$

is $(\mathscr{B}(\Delta) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(\mathbf{R}^d))$-measurable.

(ii) *For fixed* $(t_0,(v,\eta))\in[0,\infty)\times M_2$, *the solution*

$$[t_0,\infty)\times\Omega\ni(t,\omega)\mapsto x^{t_0,(v,\eta)}(t,\omega)\in\mathbf{R}^d$$

is $(\mathscr{F}_t)_{t\geqslant t_0}$*-adapted.*

Proof. Fix $\omega\in\Omega,(t_0,(v,\eta))\in[0,\infty)\times M_2, T\geqslant t_0$, till further notice. Define a sequence of functions $x_n(\cdot,\omega)\equiv x_n^{t_0,(v,\eta)}(\cdot,\omega):[t_0-r,\infty)\to\mathbf{R}^d, n\geqslant 1$, inductively via the successive approximations:

$$x_1(t,\omega)=\begin{cases} v, & t\geqslant t_0,\\ \eta(t-t_0), \text{ a.e.} & t\in[t_0-r,t_0); \end{cases}\tag{2.4}$$

$$x_{n+1}(t,\omega)=\begin{cases} \psi(t,[\zeta(t_0,v,\omega)+\int_{t_0}^t F(u,\zeta(u,x_n(u,\omega),\omega),\\ \quad x_n(u,\omega),x_{n,u}(\cdot,\omega),\omega)\,\mathrm{d}u],\omega),\ t\geqslant t_0,\\ \eta(t-t_0), \quad \text{a.e. } t\in[t_0-r,t_0) \end{cases}\tag{2.5}$$

for $n\geqslant 1$.

Clearly $x_n(\cdot,\omega)|[t_0,T]\in C^0([t_0,T],\mathbf{R}^d)$ for each $n\geqslant 1$. We will show that the family $\{x_n(\cdot,\omega)|[0,T]:n\geqslant 1\}$ is uniformly bounded and equicontinuous. To do this, consider the auxiliary sequence

$$\zeta_n(t,\omega):=\psi(t,\cdot,\omega)^{-1}(x_n(t,\omega)),\quad t_0\leqslant t\leqslant T,\ n\geqslant 1,$$

and rewrite the first equation in (2.5) in the form;

$$\begin{aligned}&\zeta_{n+1}(t,\omega)\\ &\quad=\psi(t_0,\cdot,\omega)^{-1}(v)+\int_{t_0}^t F(u,\zeta_n(u,\omega),x_n(u,\omega),\ x_{n,u}(\cdot,\omega),\omega)\,\mathrm{d}u,\ t\in[t_0,T]\end{aligned}\tag{2.6}$$

for $n\geqslant 1$. We claim that there are positive random constants $K_i(\omega), i=1,2$, independent of n, but depending on $\omega, t_0,(v,\eta),T$, such that

$$|\zeta_n(t,\omega)|\leqslant K_1(\omega)e^{K_2(\omega)(t-t_0)},\quad t_0\leqslant t\leqslant T,\ n\geqslant 1.\tag{2.7}$$

Let H be globally bounded by $M>0$, and let $K(\omega)>0$ be a positive random constant such that

$$\sup_{0\leqslant t\leqslant T}||[D\psi(t,v,\omega)]^{-1}||\leqslant K(\omega)(1+|v|),\quad v\in\mathbf{R}^d\tag{2.8}$$

([M-S.3], Theorem 2). Using (2.6), the global boundedness of H, and (2.8), we obtain

$$|\zeta_{n+1}(t,\omega)| \leqslant |\psi(t_0,\cdot,\omega)^{-1}(v)| + MK(\omega)\int_{t_0}^{t}(1+|\zeta_n(u,\omega)|)\,\mathrm{d}u, \quad t\in[t_0,T] \tag{2.9}$$

for $n\geqslant 1$. Define $\tilde{\zeta}_n(t,\omega) := \max(1,|\zeta_n(t,\omega)|)$, $t\in[t_0,T]$, $n\geqslant 1$. Then (2.9) implies that

$$\tilde{\zeta}_{n+1}(t,\omega) \leqslant K_1(\omega) + K_2(\omega)\int_{t_0}^{t}\tilde{\zeta}_n(u,\omega)\,\mathrm{d}u, \quad t\in[t_0,T],\ n\geqslant 1, \tag{2.10}$$

where $K_1(\omega) := \max[1, \sup_{t_0\leqslant t\leqslant T}|\psi(t,\cdot,\omega)^{-1}(v)| + MK(\omega)(T-t_0)]$ and $K_2(\omega) := MK(\omega)$. In order to establish our claim (2.7), we will actually show by induction on n that

$$\tilde{\zeta}_n(t,\omega) \leqslant K_1(\omega)e^{K_2(\omega)(t-t_0)}, \quad t_0\leqslant t\leqslant T,\ n\geqslant 1. \tag{2.11}$$

It follows immediately from the definition of $K_1(\omega)$ that $\tilde{\zeta}_1(t,\omega)\leqslant K_1(\omega)$. Hence (2.11) holds for $n=1$. Suppose now that (2.11) holds for some integer $n\geqslant 1$. Then from (2.10), we get the following relations

$$\begin{aligned}\tilde{\zeta}_{n+1}(t,\omega) &\leqslant K_1(\omega) + K_2(\omega)\int_{t_0}^{t}K_1(\omega)e^{K_2(\omega)(u-t_0)}\,\mathrm{d}u\\ &= K_1(\omega)e^{K_2(\omega)(t-t_0)}\end{aligned}$$

for $t_0\leqslant t\leqslant T$. Therefore (2.11) holds for $n+1$, and hence for all $n\geqslant 1$ by induction. Now (2.7) follows directly from (2.11); indeed one has

$$\sup_{\substack{t_0\leqslant t\leqslant T\\ n\geqslant 1}} |\zeta_n(t,\omega)| \leqslant K_1(\omega)e^{K_2(\omega)(T-t_0)} := K_3(\omega) < \infty. \tag{2.12}$$

Therefore,

$$\begin{aligned}\sup_{\substack{t_0\leqslant t\leqslant T\\ n\geqslant 1}} |x_n(t,\omega)| &= \sup_{\substack{t_0\leqslant t\leqslant T\\ n\geqslant 1}} |\psi(t,\zeta_n(t,\omega),\omega)|\\ &\leqslant K(\omega)\left[1+\sup_{\substack{t_0\leqslant t\leqslant T\\ n\geqslant 1}}|\zeta_n(t,\omega)|^2\right]\\ &\leqslant K(\omega)[1+K_3(\omega)^2] < \infty\end{aligned} \tag{2.13}$$

([Ku], p. 163; [M-S.3], Theorem 1). This shows that the family $\{x_n(\cdot,\omega)|[0,T] : n\geqslant 1\}$ is uniformly bounded. To prove its equicontinuity, let $t_0\leqslant t_1\leqslant t_2\leqslant T$, $n\geqslant 1$.

Consider the inequalities

$$\begin{aligned}|\zeta_{n+1}(t_2,\omega)-\zeta_{n+1}(t_1,\omega)| &= \left|\int_{t_1}^{t_2} F(u,\zeta_n(u,\omega),x_n(u,\omega),x_{n,u}(\cdot\,,\omega),\omega)\,\mathrm{d}u\right| \\ &\leqslant MK(\omega)\int_{t_1}^{t_2}(1+|\zeta_n(u,\omega)|)\,\mathrm{d}u \\ &\leqslant MK(\omega)[1+K_3(\omega)]|t_2-t_1|, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}|x_{n+1}(t_2,\omega)-x_{n+1}(t_1,\omega)| &= |\psi(t_2,\zeta_{n+1}(t_2,\omega),\omega)-\psi(t_1,\zeta_{n+1}(t_1,\omega),\omega)| \\ &\leqslant |\psi(t_2,\zeta_{n+1}(t_2,\omega),\omega)-\psi(t_1,\zeta_{n+1}(t_2,\omega),\omega)| \\ &\quad + |\psi(t_1,\zeta_{n+1}(t_2,\omega),\omega)-\psi(t_1,\zeta_{n+1}(t_1,\omega),\omega)| \\ &\leqslant \sup_{\substack{v\in\mathbf{R}^d\\ |v|\leqslant K_3(\omega)}} |\psi(t_2,v,\omega)-\psi(t_1,v,\omega)| \\ &\quad + \sup_{\substack{|v|\leqslant K_3(\omega)\\ t_0\leqslant t\leqslant T}} ||D_2\psi(t,v,\omega)||\cdot|\zeta_{n+1}(t_2,\omega)-\zeta_{n+1}(t_1,\omega)|. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) gives

$$\begin{aligned}&|x_{n+1}(t_2,\omega)-x_{n+1}(t_1,\omega)| \\ &\quad\leqslant \sup_{\substack{v\in\mathbf{R}^d\\ |v|\leqslant K_3(\omega)}} |\psi(t_2,v,\omega)-\psi(t_1,v,\omega)| \\ &\qquad + \sup_{\substack{|v|\leqslant K_3(\omega)\\ t_0\leqslant t\leqslant T}} ||D_2\psi(t,v,\omega)||\cdot MK(\omega)(1+K_3(\omega))|t_2-t_1|. \end{aligned} \tag{2.16}$$

Using the fact that the map $[t_0,T]\ni t\mapsto\psi(t,\cdot\,,\omega)\in C^0(\mathbf{R}^d,\mathbf{R}^d)$ is *uniformly* continuous (into the toplogy of uniform convergence on compacta), it follows easily from (2.16) that the family $\{x_n(\cdot\,,\omega)|[t_0,T]:n\geqslant 1\}$ is equicontinuous. By Ascoli-Arzela's lemma, this family is relatively compact in $C^0([t_0,T],\mathbf{R}^d)$, given the supremum norm. By continuity of $F(\cdot\,,\cdot\,,\cdot\,,\cdot\,,\omega)$ and $\psi(t,\cdot\,,\omega)$, it follows from (2.5) and the dominated convergence theorem that every convergent subsequence of $\{x_n(\cdot\,,\omega)\}_{n=1}^{\infty}$ converges (uniformly) to the unique solution $x(\cdot\,,\omega):[t_0-r,T]\to\mathbf{R}^d$ of (III). Hence the full sequence $\{x_n(\cdot\,,\omega)\}_{n=1}^{\infty}$ must converge uniformly to $x(\cdot\,,\omega):[t_0-r,T]\to\mathbf{R}^d$. More explicitly, we will denote the solution of (III) thus obtained by $x^{t_0,(v,\eta)}(\cdot\,,\omega):=x(\cdot\,,\omega)$ for $(t_0,(v,\eta))\in[0,\infty)\times M_2,\omega\in\Omega$. We contend that this solution satisfies the measurability assertions (i) and (ii) of the lemma. We will first show by induction on $n \geqslant 1$ that each random field

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v, \eta), \omega) \mapsto x_n^{t_0,(v,\eta)}(t, \omega) \in \mathbf{R}^d, \quad n \geqslant 1,$$

satisfies assertions (i), (ii) of the lemma. To see this, observe that the above assertion holds easily for $n = 1$. Suppose it holds for some $n \geqslant 1$. We will use (2.5) to show that it holds for $n+1$. To show this, observe that the path

$$[t_0, \infty) \ni u \mapsto (x_n^{t_0,(v,\eta)}(u, \omega), x_{n,u}^{t_0,(v,\eta)}(\cdot, \omega)) \in M_2$$

is continuous for fixed $\omega \in \Omega, (t_0, (v, \eta)) \in [0, \infty) \times M_2$. By Hypotheses $(W_{k,\delta})$, this implies that the integrand on the right-hand side of (2.5) is continuous in u for fixed $\omega \in \Omega, (t_0, (v, \eta)) \in [0, \infty) \times M_2$, is $(\mathscr{F}_u)_{u \geqslant t_0}$-adapted and $(\mathscr{B}(\Delta) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(\mathbf{R}^d))$-measurable in $(t_0, u, (v, \eta), \omega)$. Hence the indefinite *Riemann* integral on the right-hand side of (2.5) satisfies similar measurability properties to those of $x_n^{t_0,(v,\eta)}(t, \omega)$. Hence it follows from (2.5) that $x_{n+1}^{t_0,(v,\eta)}(t, \omega)$ also satisfies similar measurability properties to those of $x_n^{t_0,(v,\eta)}(t, \omega)$. Therefore each random field

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v, \eta), \omega) \mapsto x_n^{t_0,(v,\eta)}(t, \omega) \in \mathbf{R}^d, \quad n \geqslant 1$$

satisfies measurability assertions (i), (ii) of the lemma. Hence so does the pointwise limit $x^{t_0,(v,\eta)}(t, \omega) = \lim_{n \to \infty} x_n^{t_0,(v,\eta)}(t, \omega)$, $(t_0, t, (v, \eta), \omega) \in \Delta \times M_2 \times \Omega$. This completes the proof of Lemma 2.8. $\square$

Proof of Theorem 2.1. We will no longer suppress ω but will continue to fix $\omega \in \Omega$ till further notice. Let $\delta_1, \delta_2, \delta_3 > 0$, and $(t_0, (v, \eta)) \in [0, \delta_1] \times \bar{B}(0, \delta_2)$. By Lemmas 2.6 and 2.7, the contraction $U_{\tau_2}(t_0, (v, \eta), \cdot)$ has a unique fixed point $\bar{x}^{t_0,(v,\eta)} \in {}_0\mathscr{E}^{\tau_2}(\delta_3)$ which depends continuously on $(t_0, (v, \eta)) \in [0, \delta_1] \times \bar{B}(0, \delta_2)$. The map

$$x^{t_0,(v,\eta)}(t) := \begin{cases} v + \bar{x}^{t_0,(v,\eta)}(t - t_0), & t_0 \leqslant t \leqslant t_0 + \tau_2, \\ \eta(t - t_0), & t_0 - r \leqslant t < t_0, \end{cases}$$

is the unique local solution of (III) with $T = \tau_2$. By local uniqueness of solutions to (III), it follows that for $t_0 \leqslant t_1 \leqslant t \leqslant t_0 + \tau_2$, $t_1 \leqslant \delta_1$ and $(v, \eta) \in \bar{B}(0, \delta_2)$, we have

$$x^{t_0,(v,\eta)}(t) = x^{t_1,(\tilde{v},\tilde{\eta})}(t),$$

where $\tilde{v} = x^{t_0,(v,\eta)}(t_1)$, $\tilde{\eta} = x_{t_1}^{t_0,(v,\eta)}$. Therefore, solutions can be extended uniquely up to the minimum of $\delta_1 + \tau_2$ and the first exit time of $x_t^{t_0,(v,\eta)}$ from $\bar{B}(0, \delta_2)$ plus τ_2. Since δ_1, δ_2, and δ_3 were arbitrary we can extend solutions up to "explosion". So we define the explosion map $\tau : [0, \infty) \times M_2 \times \Omega \to (0, \infty]$ by requiring $\tau(t_0, (v, \eta), \omega)$ to be the supremum of all times $\tau_2 > t_0$ such that (III) has a solution $x^{t_0,(v,\eta)}(\cdot, \omega)$ defined on $[t_0 - r, \tau_2]$. When $\tau(t_0, (v, \eta), \omega) < \infty$, we shall define $x^{t_0,(v,\eta)}(t, \omega) = \partial$ for all

$t \geqslant \tau(t_0, (v, \eta), \omega)$. Note that in this case, we have $\lim_{t \to \tau(t_0,(v,\eta),\omega)-} |x^{t_0,(v,\eta)}(t)| = \infty$. Therefore assertions (i), (ii), (iv) and (vii) of Theorem 2.1 are obvious.

To prove the *global* uniqueness assertion in (iii), suppose the initial-value problem (III) has two solutions $x : [t_0, \tau(t_0, (v, \eta), \omega)) \to \mathbf{R}^d$, $y : [t_0, \tau^1(t_0, (v, \eta), \omega)) \to \mathbf{R}^d$ with the same initial data $[t_0, (v, \eta)) \in [0, \infty) \times M_2$ and explosion times $\tau(t_0, (v, \eta), \omega)$ and $\tau^1(t_0, (v, \eta), \omega)$, respectively. We will show that $\tau(t_0, (v, \eta), \omega) = \tau^1(t_0, (v, \eta), \omega)$ and $x(t, \omega) = y(t, \omega)$ for all $t \in [t_0, \tau(t_0, (v, \eta), \omega))$. With no loss of generality, we may assume that $\tau^1(t_0, (v, \eta), \omega) < \tau(t_0, (v, \eta), \omega)$. By local uniqueness, there exists $\tau_2 > 0$ such that $t_0 + \tau_2 < \tau^1(t_0, (v, \eta), \omega)$ and $y|[t_0, t_0 + \tau_2] = x|[t_0, t_0 + \tau_2]$. Let τ^* be the supremum of all such τ_2. We claim that $\tau^* = \tau^1(t_0, (v, \eta), \omega) - t_0$. Suppose-if possible- that $t_0 + \tau^* < \tau^1(t_0, (v, \eta), \omega)$. Since $(y(t_0 + \tau^*), y_{t_0+\tau^*}) = (x(t_0 + \tau^*), x_{t_0+\tau^*})$, then by local uniqueness, there exists $\tau_2^* > 0$ such that $t_0 + \tau^* + \tau_2^* \leqslant \tau^1(t_0, (v, \eta), \omega)$ and $y|[t_0 + \tau^*, t_0 + +\tau^* + \tau_2^*] = x|[t_0 + \tau^*, t_0 + +\tau^* + \tau_2^*]$. This contradicts the maximality of τ^*. Hence $t_0 + \tau^* = \tau^1(t_0, (v, \eta), \omega)$ and $x(t, \omega) = y(t, \omega)$ for all $t \in [t_0, \tau^1(t_0, (v, \eta), \omega))$. By continuity of x at $\tau^1(t_0, (v, \eta), \omega)$, we must have $\lim_{t \to \tau^1(t_0,(v,\eta),\omega)-} |y(t)| = \lim_{t \to \tau^1(t_0,(v,\eta),\omega)-} |x(t)| < \infty$. This contradicts the fact that $\lim_{t \to \tau^1(t_0,(v,\eta),\omega)-} |y(t)| = \infty$. Therefore, $\tau^1(t_0, (v, \eta), \omega) = \tau(t_0, (v, \eta), \omega)$ and $y(t, \omega) = x(t, \omega)$ for all $t \in [t_0, \tau(t_0, (v, \eta), \omega))$. This proves the global uniqueness assertion in (iii).

To show (v) and (vi), observe that by Lemmas 2.6 and 2.7, the semiflow

$$D(\omega) \ni (t_0, t, (v, \eta)) \mapsto X(t_0, t, (v, \eta), \omega) := (x^{t_0,(v,\eta)}(t, \omega), x_t^{t_0,(v,\eta)}(\cdot, \omega)) \in M_2$$

has the property that for any fixed $(t_0, (v, \eta)) \in [0, \infty) \times M_2$, there exist a neighborhood $J_0 \times V$ of $(t_0, (v, \eta))$ and a $\tau' > t_0$ such that $X(t_0', t, (v', \eta'), \omega) \in M_2$ for all $(t_0', t, (v', \eta')) \in [(J_0 \times [t_0, \tau']) \cap \Delta] \times V$ and is jointly continuous in $(t_0', (v', \eta')) \in J_0 \times V$ for each $t \in [t_0, \tau']$. Denote by $\tau^*(t_0, (v, \eta), \omega)$ the supremum of all $\tau' > 0$ satisfying the above property. We will show that $\tau^*(t_0, (v, \eta), \omega) = \tau(t_0, (v, \eta), \omega)$. This will imply that $D(\omega)$, $D_{t_0,t}(\omega)$ are open subsets of $\Delta \times M_2$ and M_2, (respectively), $X(\cdot, \omega)$ is continuous on $D(\omega)$ and the explosion map $\tau(t_0, (v, \eta), \omega)$ is lower semicontinuous in $(t_0, (v, \eta))$. From the definition of $\tau^*(t_0, (v, \eta), \omega)$, it is clear that $\tau^*(t_0, (v, \eta), \omega) \leqslant \tau(t_0, (v, \eta), \omega)$. Suppose-if possible- that $\tau^*(t_0, (v, \eta), \omega) < \tau(t_0, (v, \eta), \omega)$. Then $X(t_0, \tau^*(t_0, (v, \eta), \omega), (v, \eta), \omega) := (x^{t_0,(v,\eta)}$ $(\tau^*(t_0, (v, \eta), \omega), \omega), x^{t_0,(v,\eta)}_{\tau^*(t_0,(v,\eta),\omega)}(\cdot, \omega))$ belongs to M_2, where $x^{t_0,(v,\eta)} : [t_0 - r,$ $\tau(t_0, (v, \eta), \omega)) \to \mathbf{R}^d$ is the maximal solution of (III). Since $X(t_0, t, (v, \eta), \omega)$ is continuous at $t = \tau^*(t_0, (v, \eta), \omega)$, we may pick any $\delta_2' > 0$ and choose a small $\varepsilon_0 = \varepsilon_0(\delta_2') > 0$ such that $X(t_0, t, (v, \eta), \omega) \in B(X(t_0, \tau^*(t_0, (v, \eta), \omega), (v, \eta), \omega), \delta_2')$ for all $t \in (\tau^*(t_0, (v, \eta), \omega) - \varepsilon_0, \tau^*(t_0, (v, \eta), \omega)]$. Now apply Lemmas 2.6 and 2.7 to get a continuous local semiflow $\tilde{X}$ on $(\tau^*(t_0, (v, \eta), \omega) - \varepsilon_0, \tau^*(t_0, (v, \eta), \omega)) \times B(X(t_0, \tau^*(t_0, (v, \eta), \omega), (v, \eta), \omega), \delta_2')$ which is defined at $(t_0'', t'', (v'', \eta'')$ for all $t_0'' \in (\tau^*(t_0, (v, \eta), \omega) - \varepsilon_0,\ \tau^*(t_0, (v, \eta), \omega)), (v'', \eta'') \in B(X(t_0, \tau^*(t_0, (v, \eta), \omega),\ (v, \eta), \omega),$ $\delta_2'), t'' \in (t_0'', t_0'' + \tau_2')$. Note that $\tau_2' = \tau_2'(\delta_2', \varepsilon_0) > 0$ depends only on $(\delta_2', \varepsilon_0)$.

Choose $0<\varepsilon<\tau_2'$. By definition of $\tau^*(t_0,(v,\eta),\omega)$, pick $\tau'\in(\tau^*(t_0,(v,\eta),\omega)-\varepsilon,\tau^*(t_0,(v,\eta),\omega))$ and a neighborhood $J'\times V'$ of $(t_0,(v,\eta))$ such that $X(t_0',t,(v',\eta'),\omega)$ is continuous in $(t_0',(v',\eta'))\in J'\times V'$ for all $t\in[t_0,\tau']$, and $X(t_0',\tau',(v',\eta'),\omega)\in B(X(t_0,\tau^*(t_0,(v,\eta),\omega),(v,\eta),\omega),\delta_2')$ for all $(t_0',(v',\eta'))\in J'\times V'$. Define

$$Y(t_0',t,(v',\eta'),\omega):=\tilde{X}(\tau',t,X(t_0',\tau',(v',\eta'),\omega),\omega)$$

for all $t_0'\in J',(v',\eta')\in V',t\in[\tau',\tau'+\tau_2']$. Using the semiflow property (vii), the above relation gives a continuous local semiflow of (III) which is defined at $\tau'+\tau_2'>\tau^*(t_0,(v,\eta),\omega)$. This contradicts the maximal property of $\tau^*(t_0,(v,\eta),\omega)$. Hence the semiflow X is continuous on $D(\omega)$. A similar argument to the above gives the $C^{k,\varepsilon}$ property of the map $D_{t_0,t}(\omega)\ni(v,\eta)\mapsto X(t_0,t,(v,\eta),\omega)\in M_2$ for fixed $(t_0,t)\in\Delta$. This proves (v) and (vi).

Finally, we prove the joint measurability properties for x and τ. To this end, $\omega\in\Omega$ *will no longer be fixed*. Fix $0\leqslant t_0<T$ and $(v,\eta)\in M_2$. By truncating, we will construct a sequence of continuous, adapted, $\mathbf{R}^d$-valued (and jointly measurable) processes $y^{t_0,(v,\eta),(N)}(t)$, $t_0-r\leqslant t\leqslant T$, with $(y^{t_0,(v,\eta),(N)}(t_0),y_{t_0}^{t_0,(v,\eta),(N)})=(v,\eta)$, and such that $y^{t_0,(v,\eta),(N)}$ converges to the solution $x^{t_0,(v,\eta)}(\cdot,\omega)$ of (III) for all $((t_0,t),(v,\eta))\in\Delta\times M_2$ and all $\omega\in\Omega$ (with respect to the compactification $\mathbf{R}^d_\partial$). Once we have shown this, then (ix) follows and hence also (viii). Assertions (vi) and (ix) together imply joint measurability of

$$\Delta\times M_2\times\Omega\ni(t_0,t,(v,\eta),\omega)\mapsto x^{t_0,(v,\eta)}(t,\omega)\in\mathbf{R}^d_\partial.$$

The joint measurability of

$$[0,\infty)\times M_2\times\Omega\ni(t_0,(v,\eta),\omega)\mapsto\tau(t_0,(v,\eta),\omega)\in(0,\infty]$$

follows directly from that of

$$\Delta\times M_2\times\Omega\ni(t_0,t,(v,\eta),\omega)\mapsto x^{t_0,(v,\eta)}(t,\omega)\in\mathbf{R}^d_\partial.$$

For each integer $N\geqslant 1$, pick a C_b^∞ function $\theta_N:M_2\to[0,1]$ such that

$$\theta_N(v,\eta)=\begin{cases}1, & ||(v,\eta)||\leqslant N,\\ 0, & ||(v,\eta)||\geqslant N+1.\end{cases}$$

Further, define $H^N(u,v,\eta):=\theta_N(v,\eta)H(u,v,\eta)$, $G^N(u,v):=\theta_N(v,0)G(u,v)$. It is easy to check that H^N and G^N also satisfy Hypotheses $(W_{k,\delta})$ and (GE)(i) of Section 3. We choose a version ψ^N of the flow of G^N which agrees with ψ up to the first exit time from $\{v:|v|\leqslant N\}$. Set $\zeta^N(t):=\psi^N(t)^{-1}$. Therefore, by (the proof of) Theorem 3.1(i) (of the next section), it follows that for every integer $N\geqslant 1$ and each $t_0\geqslant 0,(v,\eta)\in M_2$, there is a unique globally defined solution $y^{t_0,(v,\eta),(N)}:[t_0-r,\infty)\times\Omega\to\mathbf{R}^d$ of the resulting equation when H and G in (III) are replaced by H^N and G^N, respectively. The reader may note that the proof of global existence of the solution in Theorem 3.1(i) *does not depend* on the joint measurability assertions on x and τ in Theorem 2.1: The proof of Theorem 3.1(i) is a path-by-path argument. Applying Lemma 2.8 to (III) with H, G replaced by H^N, G^N, we see that for fixed $(t_0, (v, \eta)) \in [0, \infty) \times M_2$, the process $y^{t_0,(v,\eta),(N)} : [t_0 - r, \infty) \times \Omega \to \mathbf{R}^d$ is $(\mathscr{F}_t)_{t \geqslant t_0}$-adapted, and each random field

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v, \eta), \omega) \mapsto y^{t_0,(v,\eta),(N)}(t, \omega) \in \mathbf{R}^d, \quad N \geqslant 1,$$

is $(\mathscr{B}(\Delta) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(\mathbf{R}^d))$-measurable. Set $Y^N(t_0, t, (v, \eta), \omega) := (y^{t_0,(v,\eta),(N)}(t), y_t^{t_0,(v,\eta),(N)})$. Define the sequence of $(\mathscr{F}_t)_{t \geqslant t_0}$-stopping times

$$\tau^N(t_0, (v, \eta), \omega) := \inf\{t > t_0 : ||Y^N(t_0, t, v, \eta, \omega)|| > N\}$$

for $N \geqslant 1$. It is not hard to see that there is a sure event $\Omega_0 \in \mathscr{F}$ such that

$$x^{t_0,(v,\eta)}(\cdot, \omega) = y^{t_0,(v,\eta),(N)}(t, \omega), \quad t_0 < t < \tau^N(t_0, (v, \eta), \omega),$$

$$\tau(t_0, (v, \eta), \omega) = \sup_{N \geqslant 1} \tau^N(t_0, (v, \eta), \omega)$$

for all $(t_0, v, \eta, \omega) \in [0, \infty) \times M_2 \times \Omega_0$. This implies that for each $\omega \in \Omega_0$, the sequence $\{y^{t_0,(v,\eta),(N)}(t, \omega)\}_{N=1}^{\infty}$ converges to $x^{t_0,(v,\eta)}(t, \omega)$ for $((t_0, t), (v, \eta)) \in \Delta \times M_2$. This proves the joint measurability assertions for x and τ. Hence the proof of Theorem 2.1 is complete. $\square$

3. Existence of the global semiflow

The main theorem in this section gives sufficient conditions on the coefficients of the sfde (I) to guarantee that the local semiflow constructed in the last section is a global semiflow for all positive times. First, we state the following conditions:

Hypotheses (GE).

(i) For $0 < T < \infty$ there exist $C = C(T) > 0$ and $\gamma = \gamma(T) \in [0, 1)$ such that

$$|H(t, v, \eta)| \leqslant C(1 + ||(v, \eta)||_{M_2}^{\gamma})$$

for all $0 \leqslant t \leqslant T$, $(v, \eta) \in M_2$.

(ii) For each $u \geqslant 0, (v, \eta) \in M_2$, one has $H(u, v, \eta, \omega) = H(u, \eta, \omega)$. For all $0 < T < \infty$ there exists $\beta \in (0, r)$ with the property that $H(u, v, \eta, \omega) = H(u, v, \tilde{\eta}, \omega)$ whenever $0 \leqslant u \leqslant T$, and $\eta, \tilde{\eta} \in L^2([-r, 0], \mathbf{R}^d)$ are such that $\eta|[-r, -\beta] = \tilde{\eta}|[-r, -\beta]$.

(iii) For all $\omega \in \Omega$ and $0 < T < \infty$ we have $\sup_{\substack{0 \leqslant u \leqslant T \\ v \in \mathbf{R}^d}} \|(D\psi(u, v, \omega))^{-1}\| < \infty$, and there exists a positive constant $C = C(T)$ such that

$$|H(t, v, \eta)| \leqslant C(1 + \|(v, \eta)\|_{M_2})$$

for all $0 \leqslant t \leqslant T$, $(v, \eta) \in M_2$.

In the space $L(M_2)$ of all continuous linear operators $A : M_2 \to M_2$, the strong topology is the one for which all evaluations $A \to A(z), z \in M_2$, are continuous. Denote by $\mathscr{B}_s(L(M_2))$ the associated Borel σ-algebra.

Theorem 3.1. *Assume Hypotheses* $(W_{k,\delta})$ *for some* $k \geqslant 1$, $0 < \delta \leqslant 1$. *Suppose that one of the conditions* (*GE*)(*i*), (*GE*)(*ii*), (*GE*)(*iii*) *holds. Let* τ, x, *and* X *be as in Theorem* 2.1. *Then the following is true*:

(i) $\tau(t_0, (v, \eta), \omega) = \infty$ *for all* $(t_0, (v, \eta), \omega) \in \mathbf{R}^+ \times M_2 \times \Omega$.

(ii) *For each* $\omega \in \Omega$, *the map*

$$\Delta \times M_2 \ni (t_0, t, (v, \eta)) \mapsto X(t_0, t, (v, \eta), \omega) \in M_2$$

is continuous; *and for fixed* $(t_0, t, \omega) \in \Delta \times \Omega$, *the map*

$$M_2 \ni (v, \eta) \mapsto X(t_0, t, (v, \eta), \omega) \in M_2$$

is $C^{k,\varepsilon}$ *for any* $\varepsilon \in (0, \delta)$.

(iii) *For each* $\omega \in \Omega$ *and* $(t_0, t) \in \Delta$ *with* $t \geqslant t_0 + r$ *the map* $X(t_0, t, \cdot, \omega) : M_2 \to M_2$ *carries bounded sets into relatively compact sets. In particular, each Fréchet derivative* $DX(t_0, t, (v, \eta), \omega) : M_2 \to M_2$ *with respect to* $(v, \eta) \in M_2$, *is a compact linear map for* $t \geqslant t_0 + r, \omega \in \Omega$.

(iv) *The maps*

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v, \eta), \omega) \mapsto X(t_0, t, (v, \eta), \omega) \in M_2$$

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v, \eta), \omega) \mapsto DX(t_0, t, (v, \eta), \omega) \in L(M_2)$$

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v, \eta), \omega) \mapsto \|DX(t_0, t, (v, \eta), \omega)\|_{L(M_2)} \in \mathbf{R}^+$$

are $(\mathscr{B}(\Delta) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(M_2))$-*measurable*, $(\mathscr{B}(\Delta) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}_s(L(M_2)))$-*measurable*, *and* $(\mathscr{B}(\Delta) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(\mathbf{R}^+))$-*measurable respectively*.

Before we prove the theorem we quote some estimates on ψ from ([M-S.3], Theorems 1, 2), ([Ku], p. 176) and [I-S]:

$$\sup_{0 \leqslant t \leqslant T} |\psi(t, x)| \leqslant K[1 + |x|(\log^+ |x|)^{\varepsilon}], \tag{3.1}$$

$$\sup_{0 \leqslant t \leqslant T} |\zeta(t,x)| \leqslant K[1 + |x|(\log^+|x|)^\varepsilon], \tag{3.2}$$

$$\sup_{0 \leqslant t \leqslant T} ||D\psi(t,x)|| \leqslant K(1 + |x|^\varepsilon), \tag{3.3}$$

$$\sup_{0 \leqslant t \leqslant T} ||(D\psi)^{-1}(t,x)|| \leqslant K(1 + |x|^\varepsilon) \tag{3.4}$$

for all $x \in \mathbf{R}^d$, each $\varepsilon > 0$ and some $K = K(\varepsilon, \omega, T) > 0$.

The following elementary lemma will be used in the proof of Theorem 3.1. Its proof is given in Appendix A. First, we introduce some notation. Let S be a separable metric space and E be a separable Hilbert space. Denote by $C_b(S,E)$ the Banach space of all uniformly continuous globally bounded maps $f : S \to E$ given the supremum norm $||f||_\infty := \sup_{x \in S}|f(x)|$.

Lemma 3.1. *Let S be a non-empty set, and E a complete metric space. Suppose $Y_k : S \to E, k \geqslant 1$, is a sequence of maps such that for each $k \geqslant 1$, the set $Y_k(S)$ is relatively compact in E. Suppose further that the sequence $\{Y_k\}_{k=1}^\infty$ converges uniformly on S to a map $Y : S \to E$. Then $Y(S)$ is relatively compact in E.*

Proof of Theorem 3.1. We first prove assertion (i). Fix $0 \leqslant t_0 < T$, $\omega \in \Omega$, $\delta_2 > 0$, $(v,\eta) \in \bar{B}(0,\delta_2)$. Let $\tau_T(v,\eta) := \tau(t_0,(v,\eta),\omega) \wedge T$. We write $x(t) := x^{t_0,(v,\eta)}(t,\omega)$, $x_t := x_t^{t_0,(v,\eta)}(\cdot,\omega), t \geqslant t_0$. Then for $t_0 \leqslant t < \tau_T(v,\eta)$, we have

$$x(t) = \psi(t, \zeta(t, x(t))), \tag{3.5}$$

$$\zeta(t,x(t)) = \zeta(t_0,v) + \int_{t_0}^t [D\psi(u,\zeta(u,x(u)))]^{-1} H(u,x(u),x_u)\, \mathrm{d}u. \tag{3.6}$$

Suppose (GE)(i) holds. Fix $0 \leqslant t_0 < T$, $\omega \in \Omega$, and choose $\varepsilon = \frac{1-\gamma}{1+\gamma}$. For simplicity of notation, set

$$\zeta(t) := \zeta(t,x(t)), \quad \zeta^*(t) := \sup_{t_0 \leqslant u \leqslant t} |\zeta(u,x(u))| \vee 1 \tag{3.7}$$

for $t_0 \leqslant t < \tau_T(v,\eta)$. In the subsequent computations, we will denote by $K_i := K_i(T,\omega,\delta_2) > 0, i = 1,2,\ldots$, random positive constants. Then (3.6), (3.4) and (GE)(i) imply that

$$|\zeta(t)| \leqslant |\zeta(t_0)| + K_1[1 + \zeta^*(t)^\varepsilon] \int_{t_0}^t (1 + |x(u)|^\gamma + ||x_u||^\gamma)\, \mathrm{d}u \tag{3.8}$$

for all $t_0 \leqslant t < \tau_T(v,\eta)$ and all $(v,\eta) \in \bar{B}(0,\delta_2)$. It is easy to see from (3.1) and (3.5) that

$$|x(u)|^\gamma + ||x_u||^\gamma \leqslant ||\eta||^\gamma + K_2[1 + \zeta^*(u)^{(1+\varepsilon)\gamma}], \quad t_0 \leqslant u < \tau_T(v,\eta). \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\zeta^*(t) \leqslant K_3 + 1 + K_4(\zeta^*(t))^{\varepsilon} + K_5(\zeta^*(t))^{\varepsilon} \int_{t_0}^{t} [\zeta^*(u)]^{(1+\varepsilon)\gamma}\, \mathrm{d}u.$$

Dividing through by $(\zeta^*(t))^{\varepsilon}$ and using the choice of ε (viz. $1-\varepsilon = \gamma(1+\varepsilon)$), one gets

$$\zeta^*(t)^{(1-\varepsilon)} \leqslant K_6 + K_7 \int_{t_0}^{t} [\zeta^*(u)]^{(1-\varepsilon)}\, \mathrm{d}u.$$

Applying Gronwall's lemma to the above inequality, we obtain

$$\zeta^*(t) \leqslant K_8 \exp\{K_7(T-t_0)(1-\varepsilon)^{-1}\}$$

for all $t_0 \leqslant t < \tau_T(v,\eta)$ and all $(v,\eta) \in \bar{B}(0,\delta_2)$. Therefore,

$$\sup_{(v,\eta)\in \bar{B}(0,\delta_2)} \ \sup_{t_0 \leqslant t \leqslant \tau_T(v,\eta)} |\zeta^*(t)| < \infty \tag{3.10}$$

Using (3.5) and (3.1), the above inequality gives

$$\sup_{(v,\eta)\in \bar{B}(0,\delta_2)} \ \sup_{t_0 \leqslant t \leqslant \tau_T(v,\eta)} |x(t,v,\eta)| < \infty.$$

From Theorem 2.1(iv) it follows that $\tau_T(v,\eta) \equiv T$. Since T is arbitrary, we get $\tau \equiv \infty$. In particular, we have

$$\sup_{(v,\eta)\in \bar{B}(0,\delta_2)} \ \sup_{t_0 \leqslant t \leqslant T} |x(t,v,\eta)| < \infty. \tag{3.11}$$

Suppose (GE)(iii) holds. Then from (3.6), (3.5) and (3.1), we obtain the inequality

$$\zeta^*(t) \leqslant K_9 + K_{10} \int_{t_0}^{t} \zeta^*(u)[\log \zeta^*(u)]^{\varepsilon}\, \mathrm{d}u \tag{3.12}$$

for all $t_0 \leqslant t < \tau_T(v,\eta)$, all $(v,\eta) \in \bar{B}(0,\delta_2)$ and fixed $\varepsilon \in (0,1]$. Applying Gronwall's lemma to (3.12) and taking logarithms on both sides of the resulting inequality yields

$$\log \zeta^*(t) \leqslant \log K_9 + K_{10} \int_{t_0}^{t} [\log \zeta^*(u)]^{\varepsilon}\, \mathrm{d}u$$

for all $t_0 \leqslant t < \tau_T(v,\eta)$, all $(v,\eta) \in \bar{B}(0,\delta_2)$. From the above inequality, (3.11) follows because $\varepsilon \in (0,1]$.

Suppose (GE)(ii) holds. Then $H(u,\eta)$ depends only on $\eta|[-r,-\beta]$. We will use forward steps of length β to prove (3.11). Define $t_k = t_0 + k\cdot\beta$, $k \in \mathbf{N}$. Fix $\varepsilon \in (0,1)$. There exists $K_1 > 0$ such that

$$\begin{aligned} |\zeta(t, x(t,v,\eta))| &\leqslant |\zeta(t_k, x(t_k,v,\eta))| \\ &\quad + K_1\left(1 + \sup_{t_k \leqslant u \leqslant t} |\zeta(u, x(u,(v,\eta)))|^{\varepsilon}\right) \int_{t_k}^{t} |H(u, x_u(\cdot, v,\eta))|\, \mathrm{d}u \end{aligned}$$

for $k \in \mathbf{N}$ with $t_k < \tau_T(v,\eta)$ and for $t_k \leqslant t \leqslant t_{k+1}$, $t < \tau_T(v,\eta)$). By induction on k, we claim that

$$M_k := \sup_{(v,\eta)\in \bar{B}(0,\delta_2)} \int_{t_k}^{t_{k+1}} |H(u, x_u(\cdot\,, (v,\eta)))|\,\mathrm{d}u < \infty$$

for each $k \in \mathbf{N}$. By continuity of H and (GE)(ii), it is easy to see that $M_0 < \infty$. Suppose that $M_{k-1} < \infty$ for some $k \in \mathbf{N}$. Therefore,

$$\sup_{(v,\eta)\in \bar{B}(0,\delta_2)} \sup_{t_{k-1}\leqslant t \leqslant t_k} |\zeta(t)| < \infty.$$

The above estimate, together with (3.5) and (3.1), imply that

$$\sup_{(v,\eta)\in \bar{B}(0,\delta_2)} \sup_{t_{k-1}\leqslant t \leqslant t_k} |x(t)| < \infty.$$

Hence (GE)(ii) and the above estimate give $M_k < \infty$. This proves our claim. Now we can conclude that $\tau \equiv \infty$ and (3.11) holds also in this case. This proves assertion (i).

Since $\tau \equiv \infty$, assertions (ii) and the first assertion in (iv) of this theorem follow immediately from the fact that $D(\omega) = \Delta \times M_2$, assertion (vi) of Theorem 2.1, and the measurability of the random field

$$\Delta \times M_2 \times \Omega \ni (t_0, t, (v,\eta), \omega) \mapsto x^{t_0,(v,\eta)}(t,\omega) \in \mathbf{R}^d$$

in Theorem 2.1.

We now prove assertion (iii). We show that (3.11) implies that for each $t > t_0 + r$, the map $X(t_0, t, \cdot\,, \omega) : M_2 \to M_2$ maps bounded sets in M_2 to relatively compact sets. By the Arzela-Ascoli theorem, it is sufficient to prove that the family $\{x^{t_0,(v,\eta)}(\cdot\,,\omega) : (v,\eta) \in \bar{B}(0,\delta_2)\}$ is equicontinuous on $[t_0, T]$ for every $\delta_2 > 0$ and $t_0 \geqslant 0$. Let $t_0 \leqslant t < t+h \leqslant T, \delta_2 > 0, \omega \in \Omega, (v,\eta) \in \bar{B}(0,\delta_2)$. Then

$$\begin{aligned}
&|x(t+h, v, \eta) - x(t, v, \eta)| \\
&\quad = |\psi(t+h, \zeta(t+h, x(t+h, v,\eta))) - \psi(t, \zeta(t, x(t, v, \eta)))| \\
&\quad \leqslant |\psi(t+h, \zeta(t+h, x(t+h, v,\eta))) - \psi(t, \zeta(t+h, x(t+h, v, \eta)))| \\
&\qquad + |\psi(t, \zeta(t+h, x(t+h, v,\eta))) - \psi(t, \zeta(t, x(t, v, \eta)))|. \qquad (3.13)
\end{aligned}$$

From (3.10) it follows that $\sup_{(v,\eta)\in \bar{B}(0,\delta_2)} \sup_{t_0 \leqslant t < t+h \leqslant T} |\zeta(t+h, x(t+h, v, \eta))| < \infty$. In view of this, and the fact that H and $[D\psi(\cdot\,,\cdot)]^{-1}$ take bounded sets to bounded sets, it follows from (3.6) that there is a $K_{11} := K_{11}(\omega, \delta_2) > 0$ such that

$$|\zeta(t+h, x(t+h, v, \eta)) - \zeta(t, x(t, v, \eta))| \leqslant K_{11} h \qquad (3.14)$$

for $t_0 \leqslant t < t+h \leqslant T, \delta_2 > 0, \omega \in \Omega, (v,\eta) \in \bar{B}(0,\delta_2)$. Now the map $[0,T] \ni t \mapsto \psi(t, \cdot\,, \omega) \in C^k(\mathbf{R}^d, \mathbf{R}^d)$ is (uniformly) continuous (for $k \geqslant 1$) (Ku, Theorem 4.6.5). Therefore, for every bounded set $B \subset \mathbf{R}^d$, the family $\{\psi(\cdot\,, x, \omega), x \in B\}$ is equicontinuous on

$[0, T]$ and the maps $\{\psi(t, \cdot, \omega), t \in [0, T]\}$ are Lipschitz on B, uniformly with respect to $t \in [0, T]$. Let $\varepsilon > 0$. Then, using (3.13), (3.14), gives $\delta_0 := \delta_0(\varepsilon, \omega, \delta_2, T) > 0$ such that $|x(t+h, v, \eta) - x(t, v, \eta)| < \varepsilon$ for all $(v, \eta) \in \bar{B}(0, \delta_2)$ and $t_0 \leqslant t \leqslant t + h \leqslant T, 0 < h < \delta_0$. Hence the family $\{x(\cdot, (v, \eta)); (v, \eta) \in \bar{B}(0, \delta_2)\}$ is equicontinuous on $[t_0, T]$. This proves the first assertion in (iii).

We now prove that $DX(t_0, t, (v, \eta), \omega) : M_2 \to M_2$ is compact for each $(v, \eta) \in M_2, t \geqslant t_0 + r, \omega \in \Omega$. Fix $(v, \eta) \in M_2, t \geqslant t_0 + r, \omega \in \Omega$. Recall that $B(0, 1)$ is the open unit ball in M_2. We will show that $DX(t_0, t, (v, \eta), \omega)(B(0, 1))$ is relatively compact in M_2. Define the sequence of maps $Y_k(t_0, t, (v, \eta), \omega) : B(0, 1) \to M_2$, $k \geqslant 1$, by

$$Y_k(z) := k[X(t_0, t, (v, \eta) + z/k, \omega) - X(t_0, t, (v, \eta), \omega)]$$

for all $z \in B(0, 1)$, $k \geqslant 1$. Using Fréchet differentiability of $X(t_0, t, \cdot, \omega)$, it is easy to see that the sequence of maps $\{Y_k(t_0, t, (v, \eta), \omega)\}_{k=1}^{\infty}$ converges in $C_b(B(0, 1), M_2)$ to $DX(t_0, t, (v, \eta), \omega)|B(0, 1)$ (for any fixed $(t_0, t, (v, \eta), \omega)$). Since $X(t_0, t, \cdot, \omega)$ takes bounded sets to relatively compact sets, then for each $k \geqslant 1$, $Y_k(t_0, t, (v, \eta), \omega)(B(0, 1))$ is relatively compact. The relative compactness of $DX(t_0, t, (v, \eta), \omega)(B(0, 1))$ now follows from Lemma 3.1.

Finally we prove the last two measurability assertions in (iv). By separability of M_2, it is easy to see from the definition of Y_k that for each $k \geqslant 1$, the map

$$\Delta \times M_2 \times B(0, 1) \times \Omega \ni (t_0, t, (v, \eta), z, \omega) \mapsto Y_k(t_0, t, (v, \eta), \omega)(z) \in M_2$$

is jointly measurable. The strong measurability of DX follows from this by passing to the limit as $k \to \infty$ for any $(t_0, t, (v, \eta), \omega) \in \Delta \times M_2 \times \Omega$. The last measurability assertion in (iv) follows from the strong measurability of DX, the separability of M_2, and the fact that $DX(t_0, t, (v, \eta), \omega)(z)$ is continuous linear in z. The proof of Theorem 3.1 is therefore complete. $\square$

Remark. We conjecture that the linear growth condition on H (viz. $\gamma = 1$ in (GE)(i)) is sufficient for the conclusion of Theorem 3.1. Examples in [M-S.3] show that the boundedness condition on $||[D\psi(\cdot, \cdot)]^{-1}||$ is quite restrictive. The following question appears to be open: Consider a sde in $\mathbf{R}^d$ driven by finitely many Brownian motions, with globally Lipschitz diffusion coefficients and a locally Lipschitz drift satisfying a linear growth condition. Does the sde admit a strictly conservative flow in dimension $d \geqslant 2$?

4. The cocycle

In this section, we will show that, if the coefficients in (I) are time-independent, then the global semiflow in Theorem 3.1 defines a cocycle on the state space M_2. More precisely, we will require the following hypotheses.

Recall that $\bar{\mathscr{F}}$ is the P-completion of $\mathscr{F}$.

Hypotheses (C)

(i) Let $\theta: \mathbf{R} \times \Omega \to \Omega$ be a P-preserving flow on Ω, viz.
   (a) θ is $(\mathscr{B}(\mathbf{R}) \otimes \mathscr{F}, \mathscr{F})$-measurable,
   (b) $\theta(t+s, \cdot) = \theta(t, \cdot) \circ \theta(s, \cdot), \quad s, t \in \mathbf{R}$,
   (c) $\theta(0, \cdot) = I_\Omega$, the identity map on Ω,
   (d) $P \circ \theta(t, \cdot)^{-1} = P, \quad t \in \mathbf{R}$.

(ii) Let $\{\mathscr{F}_t^s : -\infty < s \leqslant t < \infty\}$ be a family of sub-σ-algebras of $\bar{\mathscr{F}}$ satisfying the following conditions:
   (a) $\theta(-r, \cdot)(\mathscr{F}_t^s) = \mathscr{F}_{t+r}^{s+r}$ for all $r \in \mathbf{R}, -\infty < s \leqslant t < \infty$.
   (b) For each $s \in \mathbf{R}$, $(\Omega, \bar{\mathscr{F}}, (\mathscr{F}_{s+u}^s)_{u \geqslant 0}, P)$ is a filtered probability space satisfying the usual conditions [Pr.2].

(iii) The Brownian motion is a *helix* with respect to θ: For every $s \in \mathbf{R}$, there exists a sure event $\Omega_s \in \mathscr{F}$ such that

$$W(t+s, \omega) = W(t, \theta(s, \omega)) + W(s, \omega)$$

for all $t \in \mathbf{R}$, all $\omega \in \Omega_s$ [Pr.1].

(iv) $H(t, v, \eta) = H(0, v, \eta)$ and $G(t, v) = G(0, v)$ for all $t \geqslant 0$, $(v, \eta) \in M_2$.

Hypotheses (C)(i),(ii),(iii) are automatically satisfied if θ is the standard two-sided shift on Wiener space. In this case, each sub-σ-algebra $\mathscr{F}_t^s, -\infty < s \leqslant t < \infty$, is generated by the increments of W in the interval $[s, t]$ (and not the values of W in $[s, t]$). By virtue of Hypotheses (C)(iii) and (C)(iv), the stochastic flow ψ of the sde (II) has a version which is a perfect cocycle over θ:

$$\psi(t_2 + t_1, \cdot, \omega) = \psi(t_2, \cdot, \theta(t_1, \omega)) \circ \psi(t_1, \cdot, \omega)$$

for all $t_1, t_2 \in \mathbf{R}^+$ and all $\omega \in \Omega$ ([A-S], [I-W]). We will adopt such a version in Theorem 4.1 below and the subsequent discussion.

Theorem 4.1. *Assume Hypotheses* $(W_{k,\delta})$ *for some* $k \geqslant 1$, $0 < \delta \leqslant 1$, *Hypotheses* (*C*), *and one of the conditions* (*GE*)(*i*), (*GE*)(*ii*), (*GE*)(*iii*). *Let* X *be as in Theorem* 3.1. *Define the map* $\hat{X}: \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ *by*

$$\hat{X}(t, (v, \eta), \omega) := X(0, t, (v, \eta), \omega)$$

for all $(t, v, \eta, \omega) \in R^+ \times M_2 \times \Omega$. *Then the following is true*:

(i) $\hat{X}(t_2 + t_1, \cdot, \omega) = \hat{X}(t_2, \cdot, \theta(t_1, \omega)) \circ \hat{X}(t_1, \cdot, \omega)$ *for all* $t_1, t_2 \in \mathbf{R}^+$ *and all* $\omega \in \Omega$.

(ii) *For each* $\omega \in \Omega$, *the map*

$$\mathbf{R}^+ \times M_2 \ni (t, (v, \eta)) \mapsto \hat{X}(t, (v, \eta), \omega) \in M_2$$

is continuous; *and for fixed* $(t, \omega) \in \mathbf{R}^+ \times \Omega$, *the map*

$$M_2 \ni (v, \eta) \mapsto \hat{X}(t, (v, \eta), \omega) \in M_2$$

is $C^{k,\varepsilon}$ for any $\varepsilon \in (0, \delta)$.

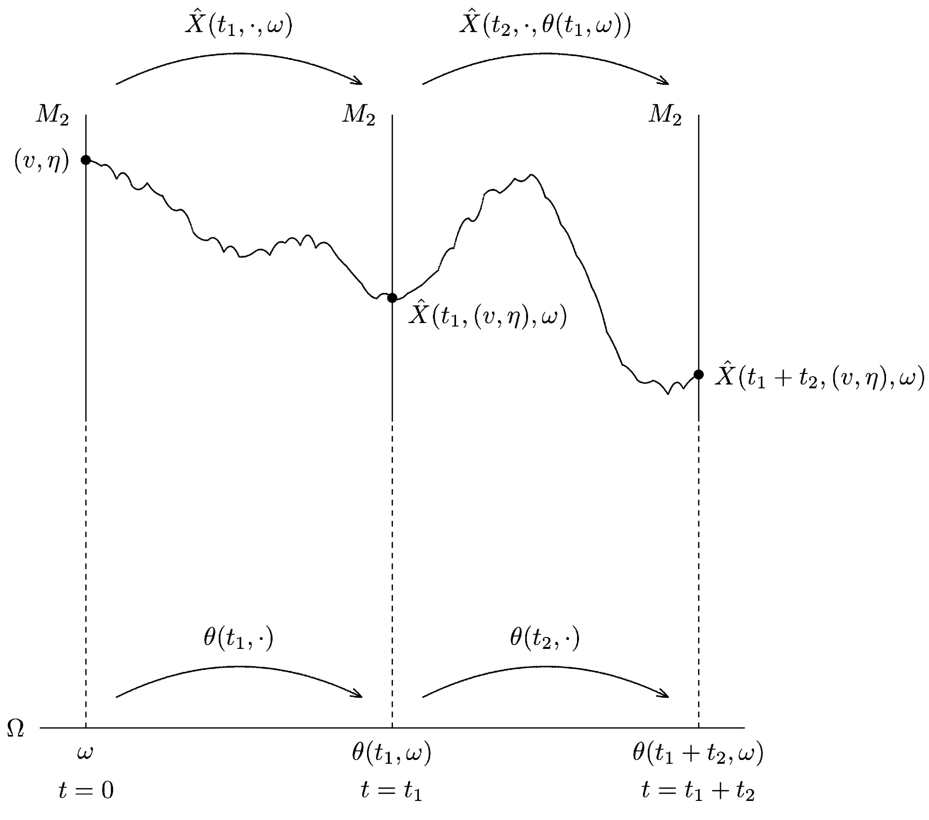
(iii) *For each* $\omega \in \Omega$ *and* $t \geqslant r$ *the map* $\hat{X}(t, \cdot, \omega) : M_2 \to M_2$ *carries bounded sets into relatively compact sets. In particular, each Fréchet derivative* $D\hat{X}(t, (v, \eta), \omega) : M_2 \to M_2$ *with respect to* $(v, \eta) \in M_2$, *is a compact linear map for* $t \geqslant r, \omega \in \Omega$.

(iv) *The maps* $\hat{X}\colon \mathbf{R}^+ \times M_2 \times \Omega \to M_2$, $D\hat{X}\colon \mathbf{R}^+ \times M_2 \times \Omega \to L(M_2)$ *are* $(\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(M_2))$*-measurable and* $(\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}_s(L(M_2)))$*-measurable, respectively. Furthermore, the function*

$$\mathbf{R}^+ \times M_2 \times \Omega \ni (t, (v, \eta), \omega) \mapsto ||D\hat{X}(t, (v, \eta), \omega)||_{L(M_2)} \in \mathbf{R}^+$$

is $(\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{B}(M_2) \otimes \mathscr{F}, \mathscr{B}(\mathbf{R}^+))$*-measurable.*

The following figure illustrates the cocycle property. The vertical solid lines represent random fibers consisting of copies of the state space M_2.



Proof of Theorem 4.1. Fix $\omega \in \Omega$ and $t_1 > 0$. To prove the cocycle property (i), it is sufficient to show that, if

$$z(t) := x^{0,(v,\eta)}(t + t_1, \omega), \quad y(t) := x^{0,\hat{X}(t_1,v,\eta,\omega)}(t, \theta(t_1, \omega)), \quad t \geqslant 0,$$

then $z(t) = y(t)$ for all $t \geqslant 0$. By equation (3.6) in the proof of Theorem 3.1, it follows that

$$\begin{aligned}&\zeta(t, y(t), \theta(t_1, \omega))\\&\quad = x^{0,(v,\eta)}(t_1, \omega) + \int_0^t [D\psi(u, \zeta(u, y(u), \theta(t_1, \omega)), \theta(t_1, \omega))]^{-1} H(u, y(u), y_u)\, \mathrm{d}u, \quad (4.1)\end{aligned}$$

and

$$\begin{aligned}&\zeta(t + t_1, z(t), \omega)\\&\quad = \zeta(t_1, v, \omega) + \int_{t_1}^{t+t_1} [D\psi(u, \zeta(u, z(u - t_1), \omega), \omega)]^{-1} H(u, z(u - t_1), z_{u-t_1})\, \mathrm{d}u \quad (4.2)\end{aligned}$$

for all $t \geqslant 0$.

Since (ψ, θ) is a cocycle, then

$$\zeta(t, z(t), \theta(t_1, \omega)) = \psi(t_1, \zeta(t + t_1, z(t), \omega), \omega), \quad t \geqslant 0. \quad (4.3)$$

Differentiating the above equation with respect to t, using the chain rule, Eq. (4.2) and Hypothesis (C)(iv), we get

$$d\,\zeta(t, z(t), \theta(t_1, \omega)) = [D\psi(t, \zeta(t, z(t), \theta(t_1, \omega)), \theta(t_1, \omega))]^{-1} H(t, z(t), z_t)\, \mathrm{d}t. \quad (4.4)$$

On the other hand, (4.1) implies

$$d\zeta(t, y(t), \theta(t_1, \omega)) = [D\psi(t, \zeta(t, y(t), \theta(t_1, \omega)), \theta(t_1, \omega))]^{-1} H(t, y(t), y_t)\, \mathrm{d}t. \quad (4.5)$$

Comparing (4.4) and (4.5), we see that z and y are solutions of the same neutral functional differential equation

$$d\zeta(t, f(t), \theta(t_1, \omega)) = [D\psi(t, \zeta(t, f(t), \theta(t_1, \omega)), \theta(t_1, \omega))]^{-1} H(t, f(t), f_t)\, \mathrm{d}t \quad (4.6)$$

with the initial condition

$$(f(0), f_0) = (z(0), z_0) = (y(0), y_0) = \hat{X}(t_1, (v, \eta), \omega). \quad (4.7)$$

The coefficients of the neutral fde (4.6) are Lipschitz on bounded sets, because the maps $\zeta(t, \cdot, \theta(t_1, \omega)), \psi(t, \cdot, \theta(t_1, \omega)), H, [D\psi(t, \zeta(t, \cdot, \theta(t_1, \omega)), \theta(t_1, \omega))]^{-1}$ are Lipschitz on bounded sets uniformly with respect to t on compacta. It is therefore easy to see that (4.6) admits a unique solution satisfying (4.7). This implies that $y(t) = z(t)$ for all $t \geqslant 0$. Hence $\hat{X}$ satisfies the cocycle property (i).

The assertions (ii), (iii), and (iv) follow immediately from the corresponding ones in Theorem 3.1. This completes the proof of Theorem 4.1. □

Remark. It is easy to see that in the preceding discussions of Sections 2–4, one can allow the initial instant t_0 to be any (negative) real number and then solve the sfde (I) (and (III)) forward in time. With this convention, and using the same proof above (replacing $\hat{X}(t_1,(v,\eta),\omega)$ with (v,η)), it follows immediately that

$$X(t_2+t_1,t_1,(v,\eta),\omega)=\hat{X}(t_2,(v,\eta),\theta(t_1,\omega))$$

for all $t_1\in\mathbf{R}, t_2\in\mathbf{R}^+$ and all $\omega\in\Omega$. Furthermore, by Theorem 3.1(iv), we see that the maps

$$\mathbf{R}\times\mathbf{R}^+\times M_2\times\Omega\ni(t_1,t_2,(v,\eta),\omega)\mapsto\hat{X}(t_2,(v,\eta),\theta(t_1,\omega))\in M_2$$

$$\mathbf{R}\times\mathbf{R}^+\times M_2\times\Omega\ni(t_1,t_2,(v,\eta),\omega)\mapsto D\hat{X}(t_2,(v,\eta),\theta(t_1,\omega))\in L(M_2)$$

are $(\mathscr{B}(\mathbf{R})\otimes\mathscr{B}(\mathbf{R}^+)\otimes\mathscr{B}(M_2)\otimes\mathscr{F},\mathscr{B}(M_2))$-measurable and $(\mathscr{B}(\mathbf{R})\otimes\mathscr{B}(\mathbf{R}^+)\otimes\mathscr{B}(M_2)\otimes\mathscr{F},\mathscr{B}_s(L(M_2)))$-measurable, respectively.

5. Extensions and generalizations

In this section, we give some extensions and generalizations of the sfde (I). We will use the following additional hypotheses:

Hypotheses ($C'_{k,\delta}$).

(i) $g:L^2([-r,0],\mathbf{R}^d)\to\mathbf{R}^m$ is $C^{k,\delta}$ $(k\geqslant 1,\delta\in(0,1])$ with Fréchet derivative Dg globally bounded.

(ii) For each $x\in L^2([-r,T],\mathbf{R}^d)$, the map $\mathbf{R}\ni t\mapsto g(x_t)\in\mathbf{R}^m$ is locally of bounded variation with square-integrable derivative $[0,T]\ni t\mapsto\frac{\mathrm{d}g(x_t)}{\mathrm{d}t}\in\mathbf{R}^m$. For each $0<T<\infty$ and $\omega\in\Omega$, the map

$$L^2([-r,T],\mathbf{R}^d)\ni x\mapsto\left\{t\mapsto\frac{\mathrm{d}g(x_t)}{\mathrm{d}t}\right\}\in L^2([0,T],\mathbf{R}^m)$$

is $C^{k,\delta}$ and globally bounded.

Examples. Let $f\in C_b^\infty(\mathbf{R}^d,\mathbf{R}^m)$, and $\sigma\in C_b^\infty(\mathbf{R}^d,\mathbf{R}^d)$ be globally bounded. Define the mapping $g:L^2([-r,0],\mathbf{R}^d)\to\mathbf{R}^m$ by

$$g(\eta):=f\left\{\int_{-r}^0\sigma(\eta(s))\,\mathrm{d}s\right\},\quad \eta\in L^2([-r,0],\mathbf{R}^d).$$

The reader may check that g is $C^{2,1}$ and satisfies Hypotheses ($C'_{1,1}$). This is because

$$\frac{dg(x_t)}{dt}=Df\left\{\int_{t-r}^t\sigma(x(u))\,\mathrm{d}u\right\}[\sigma(x(t))-\sigma(x(t-r))],\quad t\in[0,T],$$

for each $x \in L^2([-r, T], \mathbf{R}^d)$. Note the presence of the delay on the right-hand side of the above identity. The case of several discrete delays in the drift term is treated in (iii) below. The function g above is an example of a *quasitame function* ([Mo.1], Definition (4.2), p. 105). Quasitame functions can be used to give a large class of functions $L^2([-r,0], \mathbf{R}^d) \to \mathbf{R}^m$ satisfying Hypotheses $(C'_{1,1})$.

We now describe some extensions and generalizations of the sfde (I).

(i) The construction of the semiflow in this article covers a large class of regular stochastic differential systems with finite memory and random coefficients. Consider a sfde of the form

$$\left.\begin{aligned} &dx(t) = H(t, x(t), x_t)\mu(dt) + G(dt, x(t), g(x_t)), \quad t \geqslant t_0 \geqslant 0, \\ &x(t_0) = v \in \mathbf{R}^d, \quad x_{t_0} = \eta \in L^2([-r,0], \mathbf{R}^d). \end{aligned}\right\} \qquad (\mathrm{I}')$$

In the above equation, $G\colon \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^m \times \Omega \to \mathbf{R}^d$ is a Kunita-type spatial local martingale of class $B_{ub}^{k+1,\delta}$ on $\mathbf{R}^d \times \mathbf{R}^m$ (in the sense of [Ku], p. 85), and the random drift $H : [0, \infty) \times M_2 \times \Omega \to \mathbf{R}^d$ is such that for a.a. $\omega \in \Omega$ the map $H(\cdot, \cdot, \omega)$ satisfies Hypotheses $(W_{k,\delta})(1)$. Assume also that H and G are $(\mathscr{F}_t)_{(t \geqslant 0)}$-adapted and stationary in the sense that $H(t, \cdot, \omega) = H(0, \cdot, \theta(t, \omega))$, $G(t, \cdot, \omega) = G(0, \cdot, \theta(t, \omega))$ for all $\omega \in \Omega$, $t \geqslant 0$. The process $\mu(t)$, $t \geqslant 0$, is an increasing, continuous and $(\mathscr{F}_t)_{(t \geqslant 0)}$-adapted helix with $\mu(0) = 0$. Note the presence of the memory term $g(x_t)$ in the diffusion coefficient. Assume that $g : L^2([-r,0], \mathbf{R}^d) \to \mathbf{R}^m$ satisfies Hypotheses (C'). Set $h(t,x) := \frac{dg(x_t)}{dt}$. Denote $y(t) := g(x_t)$ and rewrite (I$'$) in the form

$$dx(t) = H(t, x(t), x_t)\mu(dt) + G(dt, x(t), y(t)), \quad t \geqslant t_0 \geqslant 0,$$

$$dy(t) = h(t,x)\,\mathrm{d}t, \quad t \geqslant t_0 \geqslant 0,$$

$$x(t_0) = v \in \mathbf{R}^d, \quad x_{t_0} = \eta \in L^2([-r,0], \mathbf{R}^d), \; y(t_0) = g(\eta).$$

The reader may check that the analysis in Sections 2, 3 and 4 of this article is easily adapted to handle the above sfde and hence (I$'$). In particular, (I$'$) satisfies Theorem 4.1 under the above conditions. However, see (iii) below for possible limitations on the smoothness of the semiflow. It is important to note that if the map $\mathbf{R} \ni t \mapsto g(x_t) \in \mathbf{R}^m$ is *not* of locally bounded variation, then (I$'$) may be *singular* and a stochastic semiflow will not exist even in the linear case, e.g. the one-dimensional stochastic delay equation

$$dx(t) = x(t-1)\,\mathrm{d}W(t), \quad t \geqslant 0.$$

[Mo.1,Mo.4,M-S.2].

Note further that if μ in (I') has absolutely continuous paths, then one may without loss of generality assume that $\mu(t, \omega) = t$ for all $t \geqslant 0$. Since $\mu(\cdot, \omega)$ is absolutely continuous, then we can write $d\mu(t) = \mu'(t)\,\mathrm{d}t$ where the derivative $\mu'(\cdot,\omega)$ is locally integrable. If μ has stationary increments, then μ' is a stationary process. A new drift $\tilde{H}$ can be redefined as $\tilde{H} := H \cdot \mu'$. Since H is stationary, then so is $\tilde{H}$. Therefore (I′) reduces to

$$dx(t) = \tilde{H}(t, x(t), x_t)\,\mathrm{d}t + G(dt, x(t), g(x_t)), \quad t \geqslant t_0 \geqslant 0,$$

$$x(t_0) = v \in \mathbf{R}^d, \quad x_{t_0} = \eta \in L^2([-r,0], \mathbf{R}^d).$$

(ii) Throughout this paper, we use M_2 as state space. We prefer this space because it is a Hilbert space. The Hilbert space structure is used heavily in developing the non-linear multiplicative ergodic theory in Part II (cf. [Ru]). Other choices of state space are possible; e.g. the Banach space of continuous paths $C([-r,0], \mathbf{R}^d)$ or Skorohod paths $D([-r,0], \mathbf{R}^d)$. See [M-S.1]. However, a non-linear ergodic theory for these function spaces does not seem to exist at the present time.

(iii) The case of several discrete (or distributed) delays *in the drift term* can also be treated by the methods of this article. Consider the sfde

$$\left.\begin{aligned} &dx(t) = H(t, x(t-d_m), \ldots, x(t-d_1), x(t), x_t)\mu(dt) + G(dt, x(t), g(x_t)), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geqslant t_0 \geqslant 0, \\ &x(t_0) = v \in \mathbf{R}^d, \quad x_{t_0} = \eta \in L^2([-r,0], \mathbf{R}^d), \end{aligned}\right\} \qquad (\mathrm{I}'')$$

where $d_i \in [0,r]$, $1 \leqslant i \leqslant m$, are finite delays, and $H: \mathbf{R}^+ \times \mathbf{R}^m \times M_2 \times \Omega \to \mathbf{R}^d$ and $G: \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^m \times \Omega \to \mathbf{R}^d$ are maps satisfying conditions analogous to those in (i) and Hypotheses (GE) of Section 3. Under such conditions, one gets a semiflow $X: \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ for (I″) that satisfies the conclusions of Theorem 3.1, with the following modification to the second assertion in (ii): The map

$$M_2 \ni (v,\eta) \mapsto \hat{X}(t,(v,\eta),\omega) \in M_2$$

is in general just $C^{1,1}$ (even if H and G are C_b^∞ and bounded). A similar remark holds for assertion (vi) of Theorem 2.1. To see this, simply consider the one-dimensional non-linear deterministic delay equation:

$$\left.\begin{aligned} &dx(t) = h(x(t-r))\,\mathrm{d}t, \quad t \geqslant t_0 \geqslant 0, \\ &x(t_0) = v \in \mathbf{R}, \quad x_{t_0} = \eta \in L^2([-r,0], \mathbf{R}), \end{aligned}\right\} \qquad (\mathrm{I}''')$$

where $h: \mathbf{R} \to \mathbf{R}$ is a C_b^∞ function. For $0 \leqslant t \leqslant r$, the semiflow for (I‴) is given by

$$X(t,(v,\eta))(s) = \begin{cases} v + \int_0^{t+s} h(\eta(u-r))\,\mathrm{d}u, & t+s > 0, \\ \eta(t+s), & -r \leqslant t+s \leqslant 0. \end{cases}$$

It is easy to see that for a fixed $t > 0$, the map $M_2 \ni (v,\eta) \mapsto X(t,(v,\eta))(0) \in \mathbf{R}$ is $C^{1,1}$ but not C^2, unless h is affine linear.

(iv) It appears that the results of this paper still hold if the increasing process μ in (I′) is allowed to have finite jumps.

Acknowledgments

The authors are most grateful to the anonymous referee for his careful reading of the manuscript and for his very helpful suggestions.

Appendix A

Proof of Lemma 2.5. By the hypothesis on f and differentiating under the integral sign, it follows that the map $(t,\lambda)\mapsto I(\lambda)(t)$ is C^k in the second variable, and

$$D^k I(\lambda)(\cdot)(t) = \int_0^t D^k f(u,\lambda)(\cdot)\mu(du)$$

for all $(t,\lambda)\in[0,T]\times\Lambda$. Using the above relation, it is easy to see that I is C^k and the map $\lambda\mapsto D^k I(\lambda)$ is C^ε. This shows that I is $C^{k,\varepsilon}$.

We now show that U is $C^{k,\varepsilon}$. Observe first that by composition, the map $(t,\lambda)\mapsto U(\lambda)(t)$ is C^k in λ. Hence, U is C^k. In particular,

$$DU(\lambda)(\cdot)(t) = D\psi(t,f(t,\lambda))\cdot Df(t,\lambda)$$

for all $(t,\lambda)\in[0,T]\times\Lambda$. Now use the above identity, Leibniz's rule and the hypotheses on ψ, f to conclude that the map $\lambda\mapsto D^k U(\lambda)$ is C^ε. This completes the proof of Lemma 2.5. □

Proof of Lemma 2.7. Since each map $U(\lambda,\cdot): B\to B, \lambda\in\Lambda$, is a contraction, then there is a unique map $\Lambda\ni\lambda\mapsto x(\lambda)\in B$ satisfying the identity

$$U(\lambda,x(\lambda)) = x(\lambda), \quad \lambda\in\Lambda. \tag{A.1}$$

We first prove that $x(\lambda)$ is continuous in λ. Fix $\lambda_0\in\Lambda$. For any $\lambda\in\Lambda$, the uniform contraction property implies that

$$|x(\lambda)-x(\lambda_0)|\leqslant L|x(\lambda)-x(\lambda_0)| + |U(\lambda,x(\lambda_0)) - U(\lambda_0,x(\lambda_0))|.$$

Therefore

$$|x(\lambda)-x(\lambda_0)|\leqslant\frac{1}{(1-L)}|U(\lambda,x(\lambda_0)) - U(\lambda_0,x(\lambda_0))|. \tag{A.2}$$

Since the map $\lambda\mapsto U(\lambda,x(\lambda_0))$ is continuous (for fixed λ_0), the above inequality shows that $x(\lambda)$ is continuous at λ_0. Furthermore, since U is C^ε in the first variable, (A.2) implies that x is C^ε (on bounded subsets of λ).

We next show that $x(\lambda)$ is C^1 in λ. Denote by $D_1U(\lambda, x)$ and $D_2U(\lambda, x)$ the partial Fréchet derivatives of U in the first and second variables, respectively. By the uniform contraction property, it is easy to see that $\sup_{\substack{\lambda\in\Lambda \\ x\in B}} ||D_2U(\lambda, x)|| \leqslant L < 1$. Therefore $I - D_2U(\lambda, x(\lambda)) \in GL(E)$ for all $\lambda \in \Lambda$. Define

$$f(\lambda, h) := [I - D_2U(\lambda, x(\lambda))]^{-1} D_2U(\lambda, x(\lambda))(h), \quad \lambda \in \Lambda, h \in N. \tag{A.3}$$

We will identify $f(\lambda, h)$ with the directional derivative

$$Dx(\lambda)(h) = \lim_{t\to 0} \frac{1}{t}[x(\lambda + th) - x(\lambda)], \quad \lambda \in \Lambda, h \in N.$$

To do this, set

$$\theta(t) := \frac{1}{t}[x(\lambda + th) - x(\lambda)] - f(\lambda, h), \quad t \neq 0.$$

Choose $t \neq 0$ and sufficiently small so that $\lambda + th \in \Lambda$. Using (A.1), (A.3) and the Mean-Value Theorem, it follows that

$$\theta(t) = \int_0^1 D_2U(\lambda, (1-u)x(\lambda + th) + ux(\lambda))(\theta(t))\, \mathrm{d}u + I_1 + I_2, \tag{A.4}$$

where

$$I_1 := \int_0^1 \{D_1U((1-u)(\lambda + th) + u\lambda, x(\lambda + th))(h) - D_1U(\lambda, x(\lambda))(h)\}\, \mathrm{d}u$$

and

$$I_2 := \int_0^1 \{D_2U(\lambda, (1-u)x(\lambda + th) + ux(\lambda)) - D_2U(\lambda, x(\lambda))\} f(\lambda, h)\, \mathrm{d}u.$$

Fix $\lambda \in \Lambda$ and $h \in N$, and let $\varepsilon > 0$. By continuity of D_1U, D_2U and x, there exists $\delta > 0$ such that

$$|I_i| < \varepsilon/2, \quad i = 1, 2$$

whenever $0 < t < \delta$. Therefore (A.4) implies that

$$|\theta(t)| \leqslant \varepsilon + |\theta(t)| \int_0^t ||D_2U(\lambda, (1-u)x(\lambda + th) + ux(\lambda))||\, \mathrm{d}u \tag{A.5}$$

if $0 < t < \delta$. By continuity of D_2U and x, one has

$$\lim_{t\to 0} \int_0^t ||D_2U(\lambda, (1-u)x(\lambda + th) + ux(\lambda))||\, \mathrm{d}u = 0.$$

Since $\varepsilon > 0$ is arbitrary, (A.5) and the above relation imply that $\lim_{t\to 0}\theta(t) = 0$. Hence x has a directional derivative $Dx(\lambda)(h) = f(\lambda, h)$ given by (A.3). From (A.3), it is easy to see that the map $\Lambda \ni \lambda \mapsto Dx(\lambda) = f(\lambda, \cdot) \in L(N, E)$ is continuous. Hence x is C^1. To see that x is C^k, use the relation

$$Dx(\lambda) = [I - D_2U(\lambda, x(\lambda))]^{-1} D_1U(\lambda, x(\lambda)), \quad \lambda \in \Lambda \tag{A.6}$$

and the fact that D_1U, D_2U are C^{k-1} to conclude that the map $\Lambda \ni \lambda \mapsto Dx(\lambda) \in L(N, E)$ is C^{k-1}. In fact, using (A.6), Leibniz's theorem, and the fact that U is $C^{k,\varepsilon}$, shows that the map Dx is $C^{k-1,\varepsilon}$. This completes the proof of Lemma 2.7. $\square$

Proof of Lemma 3.1. Let $Y_k : S \to E, k \geqslant 1$, be the sequence of maps given in the lemma. By completeness of E, it is sufficient to show that $Y(S)$ is totally bounded. Denote by ρ the metric on E and let $B(z, a)$ be the open ball in E, center z and radius a. Let $\varepsilon > 0$. Then by uniform convergence, there exists an integer $k := k(\varepsilon) \geqslant 1$ such that

$$\rho(Y(z), Y_k(z)) < \varepsilon/3 \tag{A.7}$$

for all $z \in S$. Since $Y_k(S)$ is totally bounded, there is a finite set $\{z_i\}_{i=1}^m \subseteq S$ such that

$$Y_k(S) \subseteq \bigcup_{i=1}^m B(Y_k(z_i), \varepsilon/3). \tag{A.8}$$

The following inclusion follows easily from (A.8), (A.7) and the triangle inequality:

$$Y(S) \subseteq \bigcup_{i=1}^m B(Y(z_i), \varepsilon). \tag{A.9}$$

Therefore $Y(S)$ is relatively compact in E. $\square$

References

[A-S] L. Arnold, M.K.R. Scheutzow, Perfect cocycles through stochastic differential equations, Probab. Th. Rel. Fields 101 (1995) 65–88.

[Ba] P.H. Baxendale, Stability and equilibrium properties of stochastic flows of diffeomorphisms, in: M.A. Pinsky, V. Wihstutz (Eds.), Diffusion Processes and Related Problems in Analysis, Vol. II, Birkhauser, Basel, 1992.

[Bi] J.-M. Bismut, A generalized formula of Itô and some other properties of stochastic flows, Z. Wahrscheinlichkeitstheorie. Verw. Geb. 55 (1981) 331–350.

[El] K.D. Elworthy, in: Stochastic Differential Equations on Manifolds, London Mathematical Society Lecture Notes Series, Vol. 70, Cambridge University Press, Cambridge, 1982.

[Fl.1] F. Flandoli, in: Regularity Theory and Stochastic Flows for Parabolic SPDEs, Stochastics Monographs, Vol. 9, Gordon and Breach Science Publishers, Yverdon, 1995.

[Fl.2] F. Flandoli, Stochastic flows for nonlinear second-order parabolic SPDE, Ann. Probab. 24 (2) (1996) 547–558.

[F-S] F. Flandoli, K.-U. Schaumlöffel, Stochastic parabolic equations in bounded domains: random evolution operator and Lyapunov exponents, Stochastics Stochastic Rep. 29 (4) (1990) 461–485.

[H] J.K. Hale, Theory of Functional Differential Equations, Springer, New York–Heidelberg–Berlin, 1977.

[I-W] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, 2nd Edition, North-Holland–Kodansha, Amsterdam, 1989.

[I-S] P. Imkeller, M.K.R. Scheutzow, On the spatial asymptotic behaviour of stochastic flows in Euclidean space, Ann. Probab. 27 (1) (1999) 109–129.

[Ku] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge University Press, Cambridge, New York, Melbourne, Sydney, 1990.

[Mo.1] S.-E.A. Mohammed, in: Stochastic Functional Differential Equations, Research Notes in Mathematics, Vol. 99, Pitman Advanced Publishing Program, Boston–London–Melbourne, 1984.

[Mo.2] S.-E.A. Mohammed, The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, Stochastics Stochastic Rep. 29 (1990) 89–131.

[Mo.3] S.-E.A. Mohammed, Lyapunov exponents and stochastic flows of linear and affine hereditary Systems, in: M. Pinsky, V. Wihstutz (Eds.), Diffusion Processes and Related Problems in Analysis, Vol. II, Birkhauser, Basel, 1992, pp. 141–169.

[Mo.4] S.-E.A. Mohammed, Stochastic differential systems with memory: theory, examples and applications, Proceedings of The Sixth Workshop on Stochastic Analysis, Geilo, Norway, July 29–August 4, 1996, Stochastic Analysis and Related Topics VI. The Geilo Workshop, 1996, in: L. Decreusefond, J. Gjerde, B. Oksendal, A.S. Ustunel (Eds.), Progress in Probability, Birkhäuser, Basel, 1998, pp. 1–77.

[M-S.1] S.-E.A. Mohammed, M.K.R. Scheutzow, Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, Part I: the multiplicative ergodic theory, Ann. Inst. Henri Poincaré, Probab. Statist. 32 (1) (1996) 69–105.

[M-S.2] S.-E.A. Mohammed, M.K.R. Scheutzow, Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales. Part II: Examples and case studies, Ann. Probab. 25 (3) (1997) 1210–1240.

[M-S.3] S.-E.A. Mohammed, M.K.R. Scheutzow, Spatial estimates for stochastic flows in Euclidean space, Ann. Probab. 26 (1) (1998) 56–77.

[M-S.4] S.-E.A. Mohammed, M.K.R. Scheutzow, The stable manifold theorem for stochastic differential equations, Ann. Probab. 27 (2) (1999) 615–652.

[Pr.1] Ph.E. Protter, Semimartingales and measure-preserving flows, Ann. Inst. Henri Poincaré, Probab. Statist. 22 (1986) 127–147.

[Pr.2] Ph.E. Protter, Stochastic Integration and Stochastic Differential Equations: A New Approach, Springer, Berlin, 1990.

[Ru] D. Ruelle, Characteristic exponents and invariant manifolds in Hilbert space, Ann. Math. 115 (2) (1982) 243–290.