Synchronization in Complex Dynamical Networks and Its Applications

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Abstract

In the past few years, the discovery of small-world and scale-free properties of many natural and artificial complex networks has stimulated increasing interest in further studying the underlying organizing principles of various complex networks. This has led to significant advances in better understanding the relationship between the topology and the dynamics of such complex networks. This paper reports our recent research work on the synchronization phenomenon in dynamical networks with small-world and scale-free connections. The economic-cycle synchronous phenomenon in the World Trade Web, a scale-free type of network, is used to illustrate an application of the network synchronization mechanism.

keywords: Synchronization, small-world, scale-free, world trade web, economic-cycle

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1 Introduction

Synchronization is a long-lasting fundamental concept and is, in fact, a universal phenomenon in all areas of science and technology [8, 30]. It indeed is a familiar phenomenon seen in our daily life, including for example fireflies flashing in unison, crickets chirping in synchrony, and heart cells beating in rhythm.

One of the most significant and interesting properties of a dynamical network is the synchronous motion of outputs of its nodes (network elements). Synchronization in coupled dynamical networks and systems has been studied for many years within a common framework based on nonlinear dynamical systems theory, due to its ubiquity in technological fields such as coupled laser systems, biochemical systems, and communication networks. Recently, synchronization in a network of coupled chaotic systems has become a topic of great interest [25, 41]. It has been observed, however, that most existing work have been concentrated on networks with completely regular topological structures such as chains, grids, lattices, and fully-connected graphs. Two typical cases are the discrete-time coupled map lattices [21]

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and the continuous-time cellular nonlinear networks [9]. The main benefit of these simple architectures is that it allows one to focus on the complexity caused by the nonlinear dynamics of the nodes, without considering additional complexity caused by the network structure itself. Another appealing feature is the possibility of realizing such regularly coupled networks by integrated circuits, with the obvious advantage of compactness, for potential engineering and technological applications.

The topology of a network often affects its functional behaviors. For instance, the topology of a social network affects the spreading of information and also disease, while an unsuitable topology of a power grid can damage its robustness and stability. In fact, we are confronting all kinds of networks with complex structures everyday, where handy instances are the Internet and the World Wide Web. We are particularly interested in the question of how to model such complex networks. Traditionally, the topology of a complex network is described by a completely random graph, the so-called Erdös-Rényi (*ER*) *random model* [12], which is at the opposite end of the spectrum from a completely regular network, and it is one of the oldest and perhaps also the best tool for study. However, with the increasing interest in trying to understand the essence of various real-life complex networks such as the Internet, the World Wide Web, the metabolic networks, and various social and economic networks like the scientific-collaboration network and the World Trade Web, etc. [1, 11, 24, 26, 31, 33]), we have found many interesting phenomena. Most of these phenomena cannot be well described by the ER random graph theory; therefore they stimulate a strong desire of building new network models [1, 11, 22, 26, 27].

Watts, Strogatz and Newman (WSN), for example, introduced the so-called *small-world network models*, which have short average path lengths along with large clusters [26, 38, 39]. These new models show a transition from a regular network to a random network. Both the ER random graph and WSN small-world network models have a common Poisson connectivity distribution and are homogenous in nature: each node in such networks has about the same number of connections. Another significant discovery is the *scale-free* feature in a number of real-life complex networks, whose connectivity distributions are in the power-law form as first pointed out by Barabási and Albert (BA) [4, 5]. Owing to the non-homogenous topology, i.e., most nodes have very few connections and only very few nodes have many connections, scale-free networks show a unique characteristic: robustness and yet fragility in synchronization (see [3, 10, 35] for a more precise interpretation). The aforementioned features of small-world and scale-free networks have also been empirically verified to fit many real-life complex networks: they are largely clustered with a short path length, following a power-law degree distribution.

The discovery of the small-world effect and scale-free feature of most complex networks has led to a fascinating set of common problems concerning how the network structure facilitates and constrains the network behaviors. In particular, a lot of work have been concentrated on synchronization in different small-world and scale-free dynamical network models, including the small-world network of oscillators [6, 16, 36, 39], small-world neural networks [17], small-world circle map lattices [7], coupled map lattices with small-world interactions [15, 20], and scale-free networks of oscillators [23, 29, 35, 32] and maps [2, 18, 19]. Of particular interest in real applications is the synchronization phenomenon in the scale-free World Trade Web studied [24], which will be further introduced in this article below.

The current studies of small-world and scale-free networks will continuously motivate more efforts devoted to the research on the synchronization and other aspects of various complex dynamical networks and systems.

2 Preliminaries

Consider a network of N identical, linearly and diffusively coupled nodes, in which each node is an n-dimensional continuous-time dynamical system [35, 36]:

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + c\sum_{j=1}^{N} a_{ij}\,\Gamma\mathbf{x}_j, \qquad i = 1, 2, \cdots, N, \tag{1}$$

where $\mathbf{x}_i = [x_{i1},\, x_{i2},\, \ldots,\, x_{in}]^T \in \Re^n$ are the state variables of node i, the constant $c > 0$ represents the coupling strength of the network, and $\Gamma \in \Re^{n\times n}$ is a constant 0-1 matrix linking the coupled variables. The coupling matrix $A = (a_{ij}) \in \Re^{N\times N}$ represents the coupling configuration of the network: if there is a connection between node i and node j, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$ $(i \neq j)$. Moreover, $a_{ii} = -k_i$, where k_i is the degree of node i.

The coupled network (1) is said to achieve (*asymptotical*) *synchronization* if

$$x_1(t) = x_2(t) = \cdots = x_N(t) \to s(t), \text{ as } t \to \infty, \tag{2}$$

where $s(t) \in \Re^n$ can be an equilibrium point, a periodic orbit, and even a chaotic orbit, depending on the interest of study.

Suppose that the network is connected without isolate clusters. Then, the coupling matrix A is a symmetric irreducible matrix. In this case, it can be shown that zero is an eigenvalue of A with multiplicity 1 and all the other eigenvalues of A are strictly negative, denoted by

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N\,.$$

Under some mild conditions, the synchronized states

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_N(t) = \mathbf{s}(t) \tag{3}$$

are exponentially stable, if

$$c \geq \left|\bar{d}/\lambda_2\right|\,, \tag{4}$$

where $\bar{d} < 0$ is a constant determined by the dynamics of an isolate node and the inner linking structural matrix Γ [23, 35, 36].

3 Synchronization in small-world networks

Given the dynamics of an isolate node and the inner linking structural matrix, the synchronizability of the dynamical network (1), with respect to a specific coupling configuration, is said to be *strong* if the network can synchronize with a small value of the coupling strength c.

The second-largest eigenvalue of the coupling matrix in the above globally coupled network is $-N$. This implies that for any given and fixed nonzero coupling strength c, a globally coupled network will synchronize as long as its size N is large enough. On the other hand, the second-largest eigenvalue of the coupling matrix of a nearest-neighbor coupled network tends to zero as $N \to \infty$, which implies that for any given and fixed nonzero coupling strength c, a nearest-neighbor coupled network cannot synchronize if its size N is sufficiently large [35, 36].

Now, consider the dynamical network (1) with the Newman-Watts (NW) small-world connections [28]. Let λ_{2sw} be the second-largest eigenvalue of the coupling matrix A. We found that, for any given coupling strength c: (i) for any $N > |\bar{d}|/c$, there exists a critical value $\bar{p}$ such that if $\bar{p} \leq p \leq 1$ then the small-world network will synchronize; (ii) for any given $p \in (0,\ 1]$, there exists a critical value $\bar{N}$ such that if $N > \bar{N}$ then the small-world network will synchronize [36]. These results imply that the ability of achieving synchronization in a large-size nearest-neighbor coupled network can be greatly enhanced by just adding a tiny fraction of distant links, thereby making the network a small-world model. This also reveals an advantage of small-world networks for achieving synchronization, if desired.

Inequality (4) shows that, for a finite-size dynamical network in the form of (1), and under some reasonable conditions, sufficiently strong coupling will lead to synchronization of the coupled oscillators. This has already been pointed out by Wu and Chua [40] in 1995. Recently, Barahona and Pecora [6] investigated a general version of the network equation (1), in which a general output function $H(\mathbf{x}_j)$ is used to replace the linear function $\Gamma \mathbf{x}_j$ shown above. For this kind of dynamical networks, it turns out to be possible that even sufficiently strong coupling do not guarantee synchronization. This is in a direct contrast to the above-discussed "stronger coupling makes synchronization easier" phenomenon, but is under a different network configuration. In [6], an algebraic condition was presented for the local stability of the synchronized states (3): a network is *synchronizable* if $\lambda_N/\lambda_2 < \beta$, where the constant β is determined by a master stability function. For example, $\beta \approx 37.85$ for a network of coupled standard Rössler chaotic oscillators where the coupling is through the x-coordinate only. We have observed that the NW small-world approach is more efficient than the approach of deterministically adding short-range layers, and it also ensures synchronizability when random graphs are below their percolating transition. Interestingly, adding many shortcuts does not improve much the stability, but it increases the robustness. As the number of added shortcuts grows, the synchronous behaviors of small-world networks and random graphs tend to converge. This is an effective randomization of the network through an NW small-world link-adding mechanism. This randomized region is robust to edge deletion: not until over 90% are cut will the eigenratio λ_N/λ_2 changes drastically. All of these suggest that an NW small-world generating scheme, when applied to the original nearest-neighbor coupled network of low redundancy, is a plausible strategy for synchronizable networks generation.

4 Synchronization in scale-free networks: Robust and yet fragile

Now, consider the dynamical network (1) again, but with BA scale-free connections instead. We have found [35] that the second-largest eigenvalue of the corresponding coupling matrix is very close to -1, which actually is the second-largest eigenvalue of star-shaped coupled networks. This implies that the synchronizability of a scale-free network is about the same as that of a star-shaped coupled network. It may be due to the extremely inhomogeneous connectivity distribution of such networks: a few "hubs" in a scale-free network play a similar and important role as a single center in a star-shaped coupled network.

We have also investigated [35] the robustness of synchronization in a scale-free dynamical network, against either random or specific removal of a small fraction f of nodes from the network. Clearly, the removal of some nodes from network (1) will change the coupling matrix. If the second-largest eigenvalue of the coupling matrix remains unchanged, then the synchronization stability of the network will remain unchanged after such a node removal. We found that even when as many as 5% of randomly chosen nodes are removed, the second-largest eigenvalue of the coupling matrix remains almost unchanged; therefore, the ongoing synchronization is not altered. On the other hand, although the scale-free structure is particularly well-suited to tolerate random errors, it is also particularly vulnerable to deliberate attacks. In particular, we found that under an intentional attack, the magnitude of the second-largest eigenvalue of the coupling matrix decreases rapidly; for example, it almost decreases to one half of its original value in magnitude, when only 1% fraction of the highly connected nodes were removed. At a low critical threshold, e.g., $f \approx 1.6\%$, this eigenvalue abruptly changes to zero, implying that the whole network was broken into isolate clusters; as a result, the ongoing synchronization is completely destroyed. Therefore, it is very reasonable to believe that the error tolerance and attack vulnerability of synchronizability in scale-free networks are rooted in their extremely inhomogeneous connectivity patterns.

For many real networks, heterogeneity is a common trait that frequently manifests itself in the form of an scale-free distribution of connectivities. As has been argued, such heterogeneity tends to reduce the average network distance. This leads naturally to the question of whether heterogeneity in scale-free network improves synchronizability. Nishikawa *et al.* lately addressed this concerned issue [29]. They followed the general framework formulated in [6] and observed the relation among the synchronizability of a scale-free network, described by $\frac{\lambda_N}{\lambda_2}$, the network distance $\bar{\mathcal{D}}$, and the scaling exponent γ. The answer is quite interesting: The ratio $\frac{\lambda_N}{\lambda_2}$ increases as the scaling exponent γ increases. In other words, as the network becomes more heterogenous, it becomes less synchronizable.

5 De-synchronizing complex dynamical networks

Assume that all the identical nodes in network (1) are chaotic, and denote by $h_{max} = h_1 > 0$, h_2, $\cdots$, h_n, the corresponding Lyapunov exponents of each individual $n-$dimensional dynamical node. The $\bar{d}$ in (4) can be specified in the sense of local stability of synchronized states (3) [23]. Under some natural assumptions, the inequality (4) is specified in the continuous-time case as

$$c > \frac{h_{max}}{|\lambda_2|}\,,$$

and its discrete-time form is

$$\frac{1 - e^{-h_{max}}}{|\lambda_2|} < c < \frac{1 + e^{-h_{max}}}{|\lambda_N|}\,. \tag{5}$$

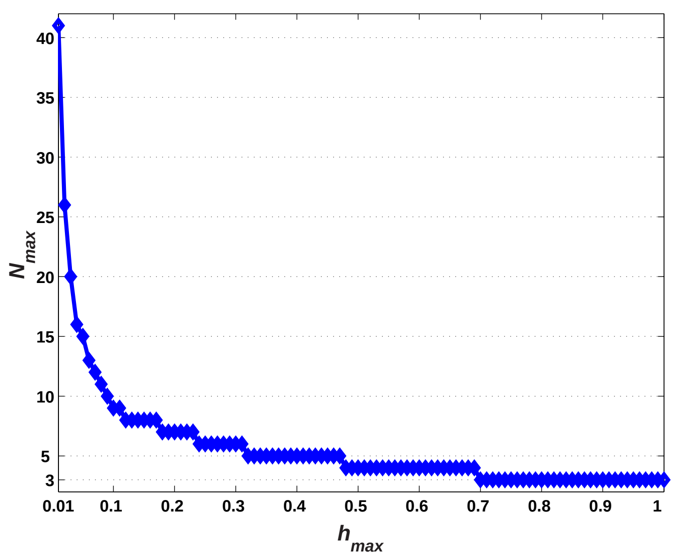

Figure 1: Distribution of the maximum scale-free network size N_{max} versus h_{max} in [0.1, 1.0] (averaged output of 20 random groups)

The existence of lower bound and upper bound of the coupling strength c in inequality (5) for synchronizing discrete-time networks is similar to the master stability synchronization in continuous-time network [6]. As an example of basic BA model for scale-free networks, with the growing mechanism of adding new nodes one by one, there exists a maximum synchronous network scale N_{max} (Fig. 1), and the network will not synchronize if the network further expands to have more than N_{max} nodes, because the condition (5) is then no longer satisfied [23]. For example, if each node is a Hénon map,

$$y(k) = -1.4y(k-1)^2 + 0.3y(k-2) + 1.0\,, \tag{6}$$

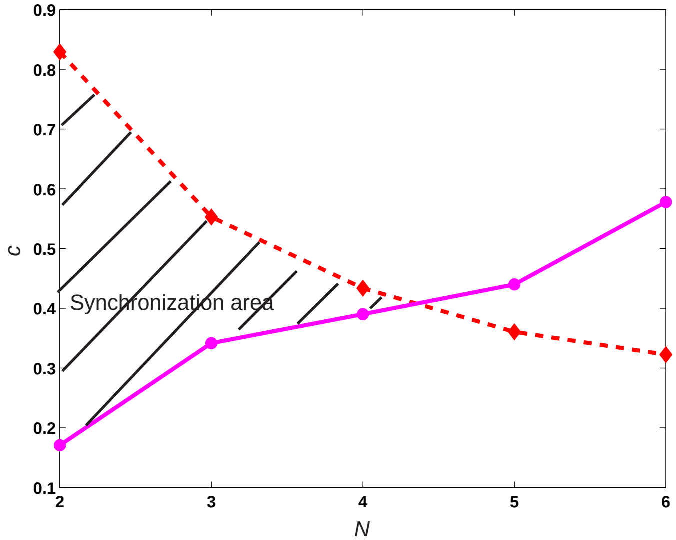

Figure 2: Synchronization area for the scale-free network of coupled Hénon maps (averaged output of 20 random groups). The dotted line is $\frac{1+e^{-h_{max}}}{|\lambda_N|}$, and the solid line is $\frac{1-e^{-h_{max}}}{|\lambda_2|}$

the maximum network scale $N_{max} = 4$ for the positive Lyapunov exponent of the map: $h_{max} = 0.418$ (Fig. 2).

Therefore, there are two conditions to be satisfied for synchronizing a discrete-time dynamical network: the network scale $N \leq N_{max}$ and the inequality (5). The network will not synchronize if either of them is not satisfied [23]. For example, as stated before, the value N_{max} of the network of coupled Hénon maps for achieving synchronization is 4. Given a 4-node network of coupled Hénon maps in a star-shaped structure, condition (5) is further changed to

$$\frac{1-e^{-h_{max}}}{|\lambda_2|} = 0.3416 < c < \frac{1+e^{-h_{max}}}{|\lambda_N|} = 0.4146\,.$$

So, if the coupling strength c is less than 0.3416 or larger than 0.4146, this 4-node network of coupled Hénon maps cannot synchronize although its network scale $N \leq N_{max} = 4$. However, if the coupling strength is suitable for synchronization, i.e., $0.3416 < c < 0.4146$, but a new Hénon node is added to the synchronous 4-node network, thereby obtaining a 5-node network (thus, $N > N_{max} = 4$), the resultant network cannot synchronize either.

6 An application example: Synchronization in the World Trade Web

Recent studies of the World Trade Web (WTW) [24, 31] have shown that there is also a significant scale-free feature in this economic network (Fig. 3). As mentioned above, the complexity of the network topology usually dominates the dynamical behaviors of a network, and the stability of a few 'big' nodes determine the synchronizability and stability of a scale-free dynamical network [35]. We have also pointed out [24] that since the United States is the biggest node in the WTW. Therefore, it is interesting to ask to what extent the economy of the United States affects the economic development in other relatively 'smaller' countries. In more subtle details, we are interested in finding whether there is synchronization of economic cycles existing between the United States and the other countries.

The economic-cycle synchronization is characterized by the correlation between the cyclical components

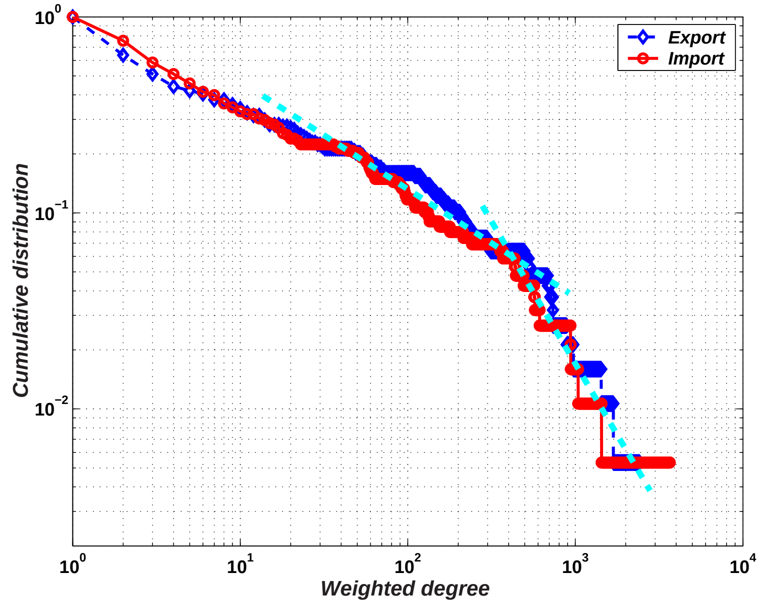

Figure 3: Cumulative import and export weighted degree distributions of the WTW with 188 nodes and 12413 export links and 12669 import links. The dashed line is the power-law form $P(k) \sim k^{-\gamma}$ with $\gamma = 0.6 \sim 1.6$

of outputs of two arbitrarily chosen countries, i and j [13, 14]:

$$corr(y_i^c, y_j^c) = \frac{cov(y_i^c, y_j^c)}{\sqrt{var(y_i^c)var(y_j^c)}}, \qquad i, j = 1, 2, \cdots, n\,, \tag{7}$$

where y_i^c is the cyclical component of output of country i, which can be specified here as the real Gross Domestic Product (GDP). A positive correlation means that synchronization exists in the economic cycles between countries i and j, and higher correlations imply a higher degree of synchronization of economic cycles.

The economy development in a country is very complex an issue, and may be affected by many unpredictable factors such as significant changes of economic or political regimes occurred in large and economically important countries (like USSR and some eastern European countries), the shocks from those speculated money firms (such as the Quantum Fund of George Soros), the continuation of a country's economic policy, the change of exchange rate policy, and local wars, etc. Hence, in studying the synchronization phenomenon of economic cycles, to decrease the perturbations on economy development from unreasonable 'noise' to the minimum, we should probe those countries where stable and peaceful political situations and mature market economic systems have been well maintained. With the statistical data of real GDP in 1975-2000 that we can collected, an appropriate choice contains 22 developed countries, i.e., the United States, the United Kingdom, Germany, Japan, France, Canada, Australia, Austria, Belgium, Norway, Italy, Finland, Denmark, Greece, Ireland, Iceland, Netherlands, Portugal, Spain, Sweden, New Zealand and Luxemburg (Fig. 4). It can be observed from the figure that in 1975-2000, totally 18 developed countries did have significant economic-cycle synchronization with the United States, where 5 countries (the United Kingdom, Australia, Canada, Finland, Sweden) show much stronger synchronization of economic cycles, except 3 countries: Japan, Germany and Austria [24].

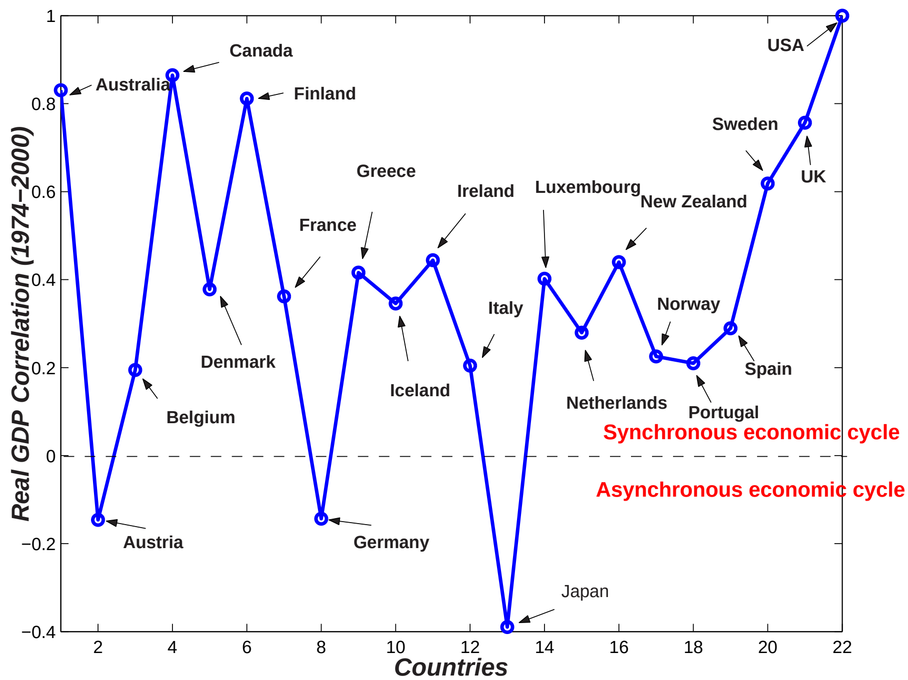

Figure 4: 22 developed countries' economic cycles synchronization phenomena. The positive real GDP correlation means synchronous economic cycles of those indicated countries with the United States, and the negative correlation means asynchronous economic cycles of the indicated countries with the United States

7 To probe further

In the past few years, advances in complex dynamical networks have uncovered some amazing similarities among such diverse systems as the Internet, cellular nonlinear networks, metabolic systems, and even the collaborations of the Hollywood movie stars. Significant progress has been gained in the studies of the effect of network topology on network dynamical behaviors, particularly the network synchronization phenomenon.

In this article, we have reported our recent research work on the synchronization of complex dynamical networks according to their different topologies with either small-world or scale-free connections. There remain plenty of important questions and technical problems about modelling, analysis, control, and synchronization of complex dynamical networks for future research. In a broader sense, the complex issues that we are facing today, from cell biology to power systems and to communication networks, all demand breakthrough ideas and revolutionary technologies. The subject of complex networks not only just poses a great challenge but also provides a great opportunity for scientists, mathematicians, physicists, and engineers alike, with foreseeable significant impacts on the modern industry, commercial markets, and beyond, therefore is worth further pursuing.

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