Estimating generating partitions of chaotic systems by unstable periodic orbits

Ruslan L. Davidchack,[1] Ying-Cheng Lai,[2] Erik M. Bollt,[3] and Mukeshwar Dhamala[1]
[1] Department of Physics and Astronomy, University of Kansas, Lawrence, Kansas 66045
[2]Department of Mathematics, Department of Electrical Engineering, Arizona State University, Tempe, Arizona 85287
[3]Department of Mathematics, 572 Holloway Road, United States Naval Academy, Annapolis, Maryland 21402
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An outstanding problem in chaotic dynamics is to specify generating partitions for symbolic dynamics in dimensions larger than 1. It has been known that the infinite number of unstable periodic orbits embedded in the chaotic invariant set provides sufficient information for estimating the generating partition. Here we present a general, dimension-independent, and efficient approach for this task based on optimizing a set of *proximity* functions defined with respect to periodic orbits. Our algorithm allows us to obtain the approximate location of the generating partition for the Ikeda-Hammel-Jones-Moloney map.

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Many chaotic systems admit a good symbolic dynamics [1]. Take, for example, the one-dimensional logistic map [2]: $x_{n+1}=rx_n(1-x_n)$. For a typical trajectory, the corresponding symbolic dynamics can be defined by associating two symbols **0** and **1** with trajectory points $0\leqslant x_n<x_c=1/2$ and $x_c<x_n\leqslant 1$, respectively. In this example, there is a one-to-one correspondence between trajectories in the phase space and itinerary sequences represented by (semi)infinite sequences of the two symbols. The critical point x_c in this case is the *generating partition* for the logistic map. For two-dimensional maps there exists no unique recipe for identifying generating partitions. For example, it is conjectured that a generating partition passes through the *primary* tangency points between the stable and unstable manifolds [3–6], which is strictly fulfilled only for specific systems such as the Hénon map [7,3,4]. For other systems, additional considerations have to be employed, such as attractor folding in the case of Duffing attractor [5], or symmetry considerations in the case of the standard map [6]. It is also possible to construct generating partitions based on a topological analysis [8]. However, this approach can only be applied to *two-dimensional* maps obtained from the Poincaré surface of section of three-dimensional flows. At present, there exists no efficient approach to identifying the generating partition for *general* high-dimensional chaotic dynamics. Being of fundamental importance to the study of chaotic dynamics, finding generating partitions is also critical for important technological applications such as communicating with chaos [9].

Given a chaotic system, it is known that the generating partition can be specified by using the set of an infinite number of unstable periodic orbits (UPO's) embedded in the underlying dynamical invariant set [10]. The general criterion is that each UPO has to be represented by a unique symbolic sequence, if the partition is generating. Thus it is possible, in principle, to assign a symbol to each UPO point in such a way that the above requirement is satisfied up to some large period p. Since the number N_p of orbit points increases exponentially as a function of the period p: $N_p\sim e^{h_T p}$, where $h_T>0$ is the topological entropy of the chaotic set, specifying a generating partition in this manner appears to be a formidable optimization problem for a large number of orbits.

In this paper, we present an efficient algorithm for estimating the location of the generating partition for a chaotic system whose attractor is dense with UPO's [11]. Our principal idea is based on the observation that the coarse features of chaotic attractors are typically revealed by a relatively small number of short UPO's, while increasingly longer orbits refine (fill-in) the features without altering the general structure. Therefore, orbit points of longer UPO's are most likely to be assigned the same symbols as the nearby points belonging to shorter UPO's. We then construct a set of *proximity* functions in the phase space, which greatly facilitates the assignment of symbols to orbit points of increasingly longer UPO's, thereby allowing us to estimate generating partitions in an extremely efficient way. Besides taking advantage of the proximity functions, our success also relies on the efficient detection of large numbers of UPO's in general chaotic systems, a task that has recently become feasible [12,13]. Our approach enables us to compute the generating partition for chaotic systems such as the Ikeda-Hammel-Jones-Moloney map [14] (see Fig. 2).

We begin by defining a *generating partition* of symbolic dynamics. The notion of *generating partition* [15] is based on the "splitting" of the phase space in terms of measurable sets [16]. Consider an N-dimensional dynamical system, $\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n)$, $\mathbf{f}{:}M\rightarrow M$. A *finite* collection of disjoint open sets, $\{B_k\}_{k=1}^K$, where $B_k\cap B_j=\varnothing$ $(k\neq j)$, is defined to be a topological partition if the union of their closures exactly covers M: $M=\cup_{k=1}^K\overline{B_k}$ [17]. Given an initial condition $\mathbf{x}_0$ and the topological partition $\{B_k\}_{k=1}^K$, the trajectory $\{\mathbf{x}_i\}_{i=-n}^n$ defines a sequence of visited partition elements: $\{B_{\mathbf{x}_i}\}_{i=-n}^n$, where $B_{\mathbf{x}_i}$ is the partition element B_k such that $\mathbf{x}_i\in B_k$. The set of the intersection of the images and preimages of these elements $\cap_{i=-n}^n\mathbf{f}^{(-i)}(B_{\mathbf{x}_i})$ is, in general, open and nonempty. For a faithful symbolic representation of the dynamics, the limit $\cap_{n=0}^\infty\cap_{i=-n}^n\mathbf{f}^{(-i)}(B_{\mathbf{x}_i})$ must be a single point. Given a dynamical system $\mathbf{f}{:}\mathcal{M}\rightarrow\mathcal{M}$ on a measure space $(\mathcal{M},F,\mu)$, a finite partition $P=\{B_k\}_{k=1}^K$ is generating if the union of all images and preimages of P gives the set of all μ-measurable sets F. In other words, the "natural" tree of partitions: $\bigvee_{i=-\infty}^\infty\mathbf{f}^{(i)}(P)$ always generates some sub-

σ-algebra, but if it gives the full σ algebra of all measurable sets F, then P is called *generating* [16]. This weaker notion says that the splitting needs only be up to measurable sets. Since our approach demands splitting of a countable set of points, the unstable periodic orbits, we do not make this distinction. For our algorithm to work, it is sufficient to demand that the UPO's be dense in the attractor, which is a common trait of chaotic attractors.

We now outline our general strategy for estimating the generating partition of chaotic systems based on the knowledge of unstable periodic orbits embedded in the chaotic set. Consider an N-dimensional dynamical system $\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n)$ and assume that we know the location of the UPO's of up to a relatively large period. The number K of symbols necessary for the symbolic representation must be large enough to allow for unique encoding of all UPO's in the system: $K \geqslant \max_p \sqrt[p]{N_p}$, where N_p is the total number of orbit points of period p, including orbit points with periods that are factors of p. Our goal is to assign each orbit point $\mathbf{x}_i$ a symbol $s_i \in \mathcal{A}=\{\alpha_1, \ldots ,\alpha_K\}$, such that all the UPO's are represented by distinct symbolic sequences. In general, this is a complicated problem of optimization for which a practical solution does not exist — a factor that hinders the determination of generating partitions from UPO's. Our success relies on the following key observation: if we start the encoding process by assigning symbols to low-period UPO's and then use them as a guidance for encoding increasingly longer UPO's, then the optimization problem becomes greatly simplified. In particular, if the encoding of short UPO's correctly reflects the overall partitioning of the phase space, then most of the orbit points of longer periods are likely to be encoded according to their *proximity* to the orbit points of shorter periods. In order to have a quantitative measure, we define the following *proximity functions of order p* for an arbitrary point $\mathbf{x}$ in the phase space:

$$Z_p^{(k)}(\mathbf{x})=\sum_{i=1}^{N_{\leqslant p}} \frac{\delta_{\alpha_k,s_i}}{|\mathbf{x}-\mathbf{x}_i|^2}, \quad k=1, \ldots ,K, \tag{1}$$

where $N_{\leqslant p}$ is the total number of orbit points whose periods are less than or equal to p, and δ_{α_k,s_i} is the Kronecker delta, which selects for the sum only those points encoded by the symbol α_k. The choice of the function $|\mathbf{x}-\mathbf{x}_i|^{-2}$ is not unique, as long as it satisfies the following requirements: it must be a positive monotone decreasing function which tends to $+\infty$ in the limit $\mathbf{x}\to\mathbf{x}_i$. In our particular choice we were primarily guided by the computational efficiency. We can now divide the phase space into K domains $\{B_k\}_{k=1}^K$ such that $Z_p^{(k)}(\mathbf{x})\geqslant Z_p^{(j)}(\mathbf{x})$, $j\neq k$, and, therefore, define a partition which distinguishes all UPO's up to at least period p.

To better illustrate the usefulness of the proximity functions, say we consider a symbolic dynamics of two symbols and the set of points on periodic orbits of periods less than or equal to p. Assume that a subset of these points: $(A_+ ,B_+ ,C_+ , \ldots)$, has already been assigned the symbol "$+$" and the complementary set $(A_- ,B_- ,C_- , \ldots)$ bears the symbol "$-$". Let $\mathbf{x}$ be a point in the phase space, then we have the following two proximity functions: $Z_p^+(\mathbf{x})=1/r_{A_+}^2 +1/r_{B_+}^2+1/r_{C_+}^2+ \ldots$ and $Z_p^-(\mathbf{x})=1/r_{A_-}^2+1/r_{B_-}^2+1/r_{C_-}^2 + \ldots$. Clearly, if $\mathbf{x}$ is "closer" to the set of "$+$" points, the distances r_{A_+}, r_{B_+}, r_{C_+}, *etc.*, are shorter, leading to $Z_p^+(\mathbf{x})>Z_p^-(\mathbf{x})$ as shown schematically in Fig. 1. While $\mathbf{x}$ can be any point on the chaotic set, choosing it to be the components of all periodic orbits of period-$(p+1)$ yields the symbolic coding for these periodic orbits, which in turn, refines the generating partition. Thus, starting from periodic orbits of the lowest period, we can, in principle, assign proper symbols to orbit points of all periods. Since unstable periodic orbits are dense on a chaotic set, in the limit $p \to\infty$, the boundaries between subsets of points with distinct symbols asymptote to the generating partition. (In Fig. 2, we see that UPO's through period-18 already "fill-in" quite well.) This strategy is powerful because the symbolic coding of the lowest periodic orbit can be readily obtained by examining the structure of the chaotic set. The method is also efficient because the computation required is just to compute the proximity functions. Occasionally, a point on a period-$(p+1)$ orbit may be assigned a wrong symbol, but this can be corrected easily by testing the uniqueness of the encoding of all period-$(p+1)$ orbits and then comparing the relative values of proximity functions at the orbit points. We find in numerical experiments that such corrections are rarely necessary.

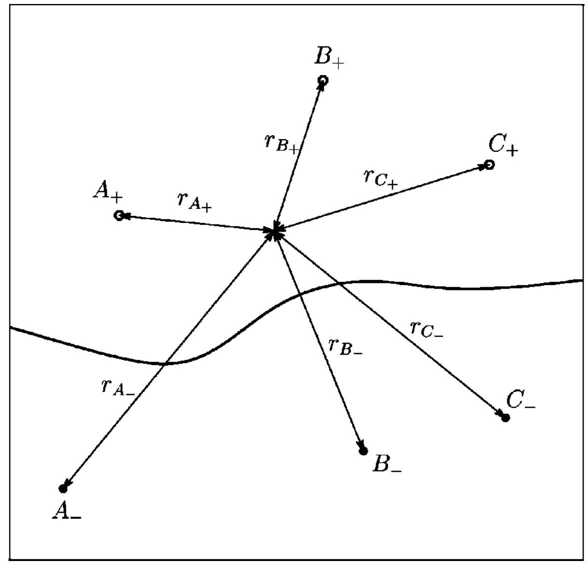

FIG. 1. A schematic illustration of the proximity function and its usage in the determination of the generating partition based on unstable periodic orbits. The two proximity functions $Z_p^+(\mathbf{x})$ and $Z_p^-(\mathbf{x})$ are equal on the curve.

Note that in the scheme just described, we do not directly search for a partitioning curve; this represents a basic difference from the existing methods [3–6]. Rather, we define a "coloring" scheme for a large list of periodic orbits, and then *we imply that a partitioning curve (or hyper surface) passes between every two unlike colored pair of points.* The gap in-between is expected to decrease with increasing period. Our approach is thus more natural than specifying an exact partitioning curve. Most importantly, once the list of periodic orbits has been found, our algorithm is essentially dimension independent.

To have confidence in our method, we tested our strategy using the Hénon map [7]: $(x,y)\to(a-x^2+by,x)$ for differ-

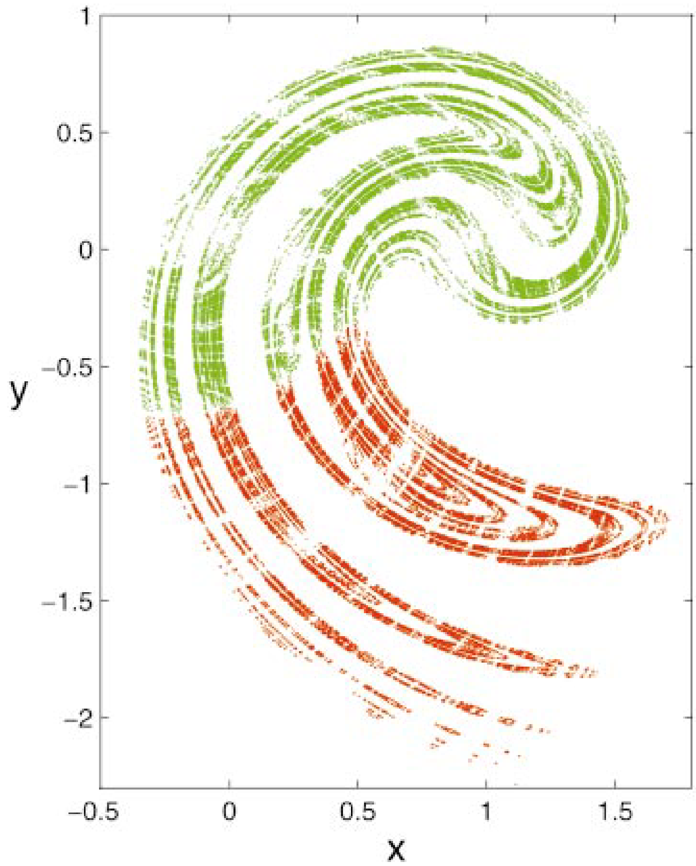

FIG. 2. (Color.) Orbit points up to period-18 for the Ikeda-Hammel-Jones-Moloney attractor in Eq. (2) colored according to their symbolic representation: green and red dots represent orbit points encoded with symbols **0** and **1**, respectively.

ent parameters (a,b), for which the generating partition has been obtained previously by examining the tangencies between stable and unstable manifolds [3]. We shall neglect the computational details here and instead, give details shortly when we compute the generating partition for the Ikeda-Hammel-Jones-Moloney map [14]. It suffices to say that generating partitions which we obtain for the Hénon map converge to those passing through the primary tangency points [3].

We now consider the Ikeda-Hammel-Jones-Moloney map [14] for which the generating partition, to our knowledge, has not been previously known. The map is given by

$$x' = a + b(x \cos\phi - y \sin\phi),$$
$$y' = b(x \sin\phi + y \cos\phi), \tag{2}$$

where $\phi = k - \eta/(1+x^2+y^2)$, and the parameters are chosen such that the map has a chaotic attractor: $a=1.0$, $b=0.9$, $k=0.4$, and $\eta=6.0$. Using a recently developed method for efficient detection of large numbers of UPO's in general chaotic systems [13] we obtain a (conjectured) complete set of UPO's of period up to 22. The topological entropy of the attractor is $h_T \approx 0.602 < \ln 2$, so the symbolic dynamics is likely to be encoded with two symbols ($\mathcal{A}=\{\mathbf{0},\mathbf{1}\}$). In the case of a binary representation it is convenient to define a single proximity function: $Z_p(\mathbf{x}) = Z_p^{(1)}(\mathbf{x}) - Z_p^{(2)}(\mathbf{x})$, which is positive in the domain of the symbol **0** and negative the domain of the symbol **1**.

We begin construction of the partition by assigning symbols to the fixed points, e.g., point (0.5328,0.2469) in the attractor is encoded with the symbol **0**, and point (1.1143, −2.2857) on the basin boundary is encoded with the symbol **1** [18]. Next, we determine the values of the proximity function $Z_1(\mathbf{x}^{(2)})$ at the positions of the two period-2 orbit points: $\mathbf{x}^{(2)} = \{(0.5098,-0.6084),(0.6216,0.6059)\}$, *relative to the just assigned partitioning of the period-1 orbits*. The corresponding values are 1.05 and 7.19, which would indicate that both points should be encoded with **0**. This encoding obviously violates the requirement for unique symbolic representation of each UPO, and, therefore, correction is necessary. An orbit point with the smallest absolute value of the proximity function is most likely to be the one encoded incorrectly. Thus the symbol representing point (0.5098, −0.6084) is changed to **1**. We then proceed to calculating $Z_{p-1}(\mathbf{x}^{(p)})$ for $p \geqslant 3$, with subsequent assignment of symbols according to the sign of the proximity functions, with uniqueness verifications after each period. In this example, no more inconsistencies are encountered until period-8, where two orbits are assigned the same sequences. The necessary correction is again obvious if we consider the values of the proximity function at the two orbits: (−554.6, −265.4, 188.3, 486.4, 664.9, 608.9, −8.4, 310.3) and (−535.2, −270.0, 188.3, 487.4, 708.1, 704.2, −98.6, 395.0). According to these values, we should change the encoding of the seventh point in the first orbit. After this correction, no more inconsistencies are detected, and we can proceed to the next period. Note that the number of corrections for this attractor is extremely small: a total of only 16 corrections was required for the encoding of all the UPO's up to period-20 (for a total of 373 005 orbit points), each correction being as simple as the one illustrated above.

The position of the generating partition for the attractor given by Eq. (2) is indicated in Fig. 2, where orbit points with periods up to 18 are colored according to their symbolic representation. The generating partition curve passes through the narrow region separating the two colors. We remark that in Fig. 2, one can see a ''shadow'' of the homoclinic tangency points [19] through our partition curve *even though no such considerations entered directly into our computations*. Apparently, the precision of thus determined generating partition can be made arbitrarily high by considering UPO's of higher period, which is, in principle, no longer an obstacle, particularly for low-dimensional chaotic systems [12,13]. We stress two facts here: (1) the generating partition obtained is only an approximation, and (2) to our knowledge, the generating partition shown in Fig. 2 for the Ikeda-Hammel-Jones-Moloney map has not been obtained previously.

Note that our approach has some common ideas with the one based on the topological analysis [8]. Both methods extract the information about symbolic dynamics progressively from the hierarchy of UPO's, and both rely on the continuity and uniqueness of the underlying dynamics. The topological analysis has the added constraint that topological invariants are compatible with symbolic names to ensure dynamic relevance. However, this analysis can only be applied to flows and hence to orientation-preserving two-dimensional maps, while our approach is simpler, more general and, in principle, dimension independent.

In summary, we have developed an efficient strategy for determining the generating partitions in chaotic systems. Our method works extremely well for low-dimensional chaotic systems for which UPO's can be readily obtained, and is applicable to high-dimensional chaotic systems as well, in so far as large numbers of UPO's can be detected.

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[18] We find that initial assignment of symbols to periodic orbits of the lowest period does not affect the final generating partition. As the period increases, the size of the phase-space region that contains the generating partition decreases rapidly.

[19] Loosely, the UPO's arrange themselves in regions of higher natural measure, which is related to the Sinai-Bowen-Ruelle (SBR) measure, or the measure along unstable manifolds (the darker colored regions in the picture). The voids, or white regions which ''stripe'' the attractor, reveal shadows of the stable manifold. A partition curve arises automatically where the white stripes and the colored stripes seem to be tangent.