Simple Connectivity and Linear Chaos

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Abstract. The goal of this paper is to show that, for domains G of the complex plane, simple connectivity can be characterized by the dynamical properties of certain linear differential operators acting on the space of functions holomorphic on G. The operator-theoretic issues that arise here lead to interesting problems, some of them apparently open. The paper is written so as to be accessible to anyone whose background includes the basics of graduate level complex and functional analysis.

Prologue

The notion of simple connectivity for plane domains stands somewhere between analysis and topology. In beginning complex analysis we learn that every nonvanishing function on such a domain has a holomorphic logarithm, and later on we encounter the Riemann Mapping Theorem, which tells us that simply connected domains are just the ones that are biholomorphically equivalent to the open unit disc. However the concept is most often defined topologically: simply connected domains are the ones in which every closed curve is null-homotopic. In fact if you open Rudin's classic analysis textbook [18] to page 247, you'll find the statements I just mentioned embedded in Theorem 13.11—a list of ten equivalent properties, each of which can be taken as the definition of simple connectivity.

The purpose of this paper is to add to these equivalences some further ones that come from the dynamical properties of linear operators. Now you may be surprised to hear that linear operators can have interesting dynamical properties, but I hope to convince you before we're done that this is indeed the case, and that such phenomena create interesting connections between operator theory, dynamics, and analytic function theory.

I've tried to make the exposition accessible to anyone with a graduate-level background in complex and functional analysis. I think of what follows as a continuation of another of my favorite books: [15] by Luecking and Rubel. If you've had the pleasure of studying from this little gem, then you should be well prepared for what lies in the pages ahead.

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1 Introduction

This work is set entirely in $\mathcal{H}(G)$, the vector space of all functions holomorphic on a plane domain G. In its natural topology—the topology of uniform convergence on compact subsets of G—this space is a complete, metrizable, locally convex linear topological space, or for short, a *Fréchet space.* A central role will be played by the quintessential linear operator on $\mathcal{H}(G)$, the operator D of (complex) differentiation. My goal here is examine how simple connectivity for G can be characterized in terms of certain dynamical properties of both D and those operators that commute with D. In order to state the results economically, I need to introduce some definitions.

1.1 Notation and terminology. Throughout this paper the symbols **C**, **N**, and **Z** denote, respectively, the complex plane, the natural numbers (i.e., the positive integers), and all the integers. The word "operator" and the phrase "linear operator" both mean "continuous linear transformation." An operator T on a Fréchet space X will be called:

- *Nonscalar* if it is not a constant multiple of the identity operator;
- *Cyclic* if there is a vector $x \in X$ (called a *cyclic vector*) whose T*-orbit*:

$$\text{Orb}(T, x) = \{T^n x : n = 0, 1, 2, \ldots\}$$

  has dense linear span in X;
- *Hypercyclic* if there is a vector $x \in X$ (called a *hypercyclic vector*) for which $\text{Orb}(T, x)$ itself is dense in X; and
- *Chaotic* [7, page 50, Defn. 8.5]* if it is hypercyclic and has a dense set of periodic points.

1.2 Notions of cyclicity. Clearly chaotic operators are hypercyclic, and hypercyclic operators are cyclic. Cyclicity is important when you study invariant subspaces (closed subspaces that get taken into themselves by the operator): an operator has a nontrivial invariant subspace precisely when it has a noncyclic vector. Hypercyclicity bears the same relationship to the existence of invariant *subsets*. You can think of hypercyclicity as a kind of topological randomness. The additional requirement that chaotic operators have dense sets of periodic points superimposes a kind of topological orderliness over all the hypercyclic randomness.

At this point you are entitled to wonder if there are any hypercyclic or chaotic operators at all! After all you've never seen such operators in linear algebra courses—for the simple reason that finite dimensional spaces do not support them (you'll will see why complex spaces don't support hypercyclic operators in §2.5). In infinitely many dimensions, however, the situation is much different, in fact some of our favorite operators turn out to be chaotic!

*However, see §1.4 before you check out this reference.

Consider, for example, the first result on hypercyclicity, obtained in 1929 by G.D. Birkhoff [2], who discovered a hypercyclic operator on the space $\mathcal{H}(\mathbf{C})$ of entire functions. Now Birkhoff's example is not something pathological, it is the beloved operator of "translation by a"

$$(1) \qquad T_a(f)(z) \stackrel{\text{def}}{=} f(z+a) \qquad (f \in \mathcal{H}(\mathbf{C}), \quad z \in \mathbf{C}),$$

where a can be any nonzero complex number. About twenty years later G.R. MacLane [16] showed that the differentiation operator D has the same property on $\mathcal{H}(\mathbf{C})$, and more recently Gilles Godefroy and I showed in [9] that *every nonscalar operator on* $\mathcal{H}(\mathbf{C})$ *that commutes with* D *is chaotic.* Thus, for example, every nonscalar constant coefficient differential operator is chaotic on $\mathcal{H}(\mathbf{C})$, as is every nontrivial linear combination of translation operators.

You may also be wondering if every hypercyclic operator has to be chaotic. The answer is "no". Elementary examples of this phenomenon occur in [9, Theorem 6.3] (see also [10, Proposition 4.7(v)] and [12] for a different construction). In [6] Kit Chan and I found that Birkhoff's original translation operators, when restricted to certain "small" Hilbert spaces of entire functions, are also hypercyclic but not chaotic.

1.3 Chaos and simple connectivity. The work in this paper was motivated by a conversation I had about eight years ago with Carl Prather of Virginia Tech. Prather asked about the fate of the results I just described when the space of entire functions is replaced by $\mathcal{H}(G)$ for G an arbitrary plane domain. In what follows I will answer Prather's question by showing that: *If G is a plane domain and P a nonconstant (holomorphic) polynomial, then the following are equivalent:*

- *$P(D)$ is hypercyclic on $\mathcal{H}(G)$.*
- *$P(D)$ is chaotic on $\mathcal{H}(G)$.*
- *G is simply connected.*

The proof of this result will occupy the next section. It turns out that the proof still works when the polynomials in D are replaced by a more extensive class of operators that commute with D. However the result does not extend to *all* operators that commute with D: In Section 3 I'll show you an infinitely connected plane domain G whose $\mathcal{H}(G)$ supports a chaotic operator that commutes with D. However if you are willing to restrict your attention solely to *finitely* connected domains, then it will turn out that the above characterization of simple connectivity goes through with polynomials in D replaced by arbitrary nonscalar operators that commute with D.

A recurrent theme in this paper is the issue of "which operators commute with differentiation?" It leads to interesting questions, some of which I discuss in (the mostly expository) Section 4.

The paper closes with a section that takes up the question of whether or not *cyclicity* can replace hypercyclicity in characterizing simple connectivity. The methods of Section 3 easily eliminate from consideration domains of genus greater than 2, but domains of genus 2—topological annuli—remain problematic. For this I present, in the hope that others may do better, a partial result that suggests an affirmative answer.

1.4 Chaotic footnote. If you look up the reference cited for the definition of "chaotic" (page 50 of Devaney's book [7]), you won't see the definition I have given in §1.1. Instead of hypercyclicity you'll see a requirement of "topological transitivity" that is phrased in terms of open sets rather than orbits. Additionally you will find a further requirement of "sensitive dependence on initial conditions." There is, however, no problem: the two definitions of "chaotic" are in fact equivalent. Topological transitivity is equivalent, for complete metric spaces, to the existence of a dense orbit (see [9, Section 1]), and in our Fréchet space setting, Devaney's notion of "sensitive dependence" turns out to be a consequence of hypercyclicity (see [9, Section 6]).

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2 Chaotic characterizations of simple connectivity

Much of the work that has been done on hypercyclic operators depends on the following sufficient condition, first discovered by Carol Kitai in her 1982 doctoral dissertation [13], but never published. Robert Gethner and I later rediscovered the result [8], and variations of it have figured prominently in much subsequent work (e.g., [4, 9, 11, 12, 20]). The proof, which is a simple Baire Category argument, occurs in many of these references, so I will omit it here, and instead refer you to Theorem 1.2 of my paper [9] with Godefroy, or to Section 7.1 of my book [20].

2.1 Theorem ("The Hypercyclicity Criterion"). *Suppose T is an operator on a Fréchet space X. Suppose further that there are dense subsets X_0 and Y_0 of X, and a mapping $S : Y_0 \to Y_0$, such that:*

(a) $T^n \to 0$ pointwise on X_0,

(b) $S^n \to 0$ pointwise on Y_0,

(c) TS is the identity map on Y_0.

Then T is hypercyclic on X.

2.2 Remark. If you have never seen this result before, don't let its apparent complexity discourage you. As you will see before too long, it is often very easy to apply. In fact, you can illustrate this for yourself right now by using the Hypercyclicity Criterion to prove Rolewicz's Theorem [17]—the first hypercyclic result for Hilbert space operators:

> *Let B denote the backward shift operator on the sequence space ℓ^2. Then for any complex number λ of modulus > 1 the operator λB is hypercyclic.*

Here B is the operator that takes an ℓ^2-sequence $(a_0, a_1, \ldots)$ to its left-shifted sequence $(a_1, a_2, \ldots)$. To prove Rolewicz's Theorem, just take X_0 to be all sequences with only finitely many nonzero terms, take Y_0 equal to ℓ^2, and set

$$S = \frac{1}{\lambda} \times \text{ the forward shift}$$

where the "forward shift" is the operator that takes $(a_0, a_1, \ldots)$ to its right-shifted sequence $(0, a_0, a_1, \ldots)$. Try it!

2.3 Non-hypercyclicity. We also need a convenient condition sufficient for an operator to *not* be hypercyclic. For this we use a result first observed by Kitai [13], which makes use of the *adjoint* of an operator on a Fréchet space. Suppose X is a Fréchet space and T is an operator on X. Let X^* denote the dual space of X—the space of continuous linear functionals on X. Then the *adjoint* of T is the linear transformation defined on X^* by the equation:

$$T^*\phi(x) = \phi(Tx) \qquad (\phi \in X^* \text{ and } x \in X).$$

Although it is possible to give X^* a topology in which T^* becomes continuous, I won't do this here: for us T^* will always be a purely algebraic object.

2.4 A "non-hypercyclicity criterion." *Suppose T is an operator on a Fréchet space X, and that T^* has an eigenvalue. Then T is not hypercyclic.*

Proof. The hypothesis states that there is a complex number λ and a continuous linear function ϕ on X, not identically zero, such that $T^*\phi = \lambda\phi$. Suppose $x \in X$. The goal is to show that x is not a hypercyclic vector for T, i.e. that the T-orbit of x is not dense in X. If x *were* hypercyclic for T then the set of values that ϕ takes on $\text{Orb}(T, x)$ would be dense in the complex plane. But instead we have for each positive integer n:

$$\phi(T^n x) \stackrel{\text{def}}{=} (T^{n*}\phi)(x) = (T^{*n}\phi)(x) = \lambda^n (T^*\phi)(x) \stackrel{\text{def}}{=} \lambda^n \phi(Tx),$$

which you can easily see shows that the set of values $\{\phi(T^n x)\}$ is not dense in X. Thus x is not hypercyclic for T, as promised. ///

2.5 Remark. I mentioned earlier that no operator on a finite dimensional space is hypercyclic. The above result shows why this is true, at least for complex vector spaces. For if T is an operator on a complex Fréchet space X of dimension $0 < n < \infty$ then, because X^* also has dimension n, T^* must have an eigenvalue. Thus T cannot be hypercyclic. Real finite dimensional Fréchet spaces require a somewhat more complicated proof, but this too is possible: it was done by Paul Bourdon [3].

2.6 Complex exponentials. For $\lambda \in \mathbf{C}$ let e_λ denote the exponential function

$$e_\lambda(z) = e^{\lambda z}.$$

Interest in these exponential functions stems from their role as eigenfunctions of the differentiation operator: $De_\lambda = \lambda e_\lambda$, and so if P is a holomorphic polynomial, then $P(D)e_\lambda = P(\lambda)e_\lambda$ for each $\lambda \in \mathbf{C}$. Particularly important are the subspaces spanned by various collections of these eigenfunctions. If A is a subset of the complex plane, let

$$\mathcal{E}(A) \stackrel{\text{def}}{=} \text{span}\,\{e_\lambda : \lambda \in A\}.$$

The following result is well known: I include it here in the interest of completeness.

2.7 Density Theorem. *If G is simply connected then $\mathcal{E}(A)$ is dense in $\mathcal{H}(G)$ whenever A has a limit point in $\mathbf{C}$.*

Proof. We use the Hahn-Banach theorem. Suppose ϕ is a continuous linear functional on $\mathcal{H}(G)$ that annihilates each exponential function e_λ for $\lambda \in A$. By Hahn-Banach we will be done if we can show that $\phi = 0$ on $\mathcal{H}(G)$. Since ϕ is continuous, the inverse image of the unit disc contains a basic neighborhood of zero in $\mathcal{H}(G)$. Now these basic neighborhoods of zero have the form

$$N(K, \varepsilon) \stackrel{\text{def}}{=} \{f \in \mathcal{H}(G) : ||f||_K < \varepsilon\}$$

where K runs through compact subsets of G, ε through positive reals, and $||f||_K$ is the maximum of $|f(z)|$ as z runs through K.

Thus there is a compact subset K of G and a positive number ε such that $|\phi(f)| \leq 1$ whenever $||f||_K < \varepsilon$, i.e.,

$$|\phi(f)| \leq \frac{1}{\varepsilon}||f||_K \qquad (\text{all } f \in \mathcal{H}(G)) \tag{2}$$

Now we may, without loss of generality, suppose that K has nonempty interior (since the above inequality just gets better if K is larger), so that $\mathcal{H}(G)$ can be injectively regarded, via restriction to K, as a (nonclosed) subspace of $C(K)$. Inequality (2) asserts that, for the $C(K)$-norm, ϕ is continuous on this subspace, so by the Hahn-Banach Theorem it extends to a continuous linear functional $\overline{\phi}$ on $C(K)$ itself. The Riesz Representation Theorem provides a complex Borel measure μ on K such that

$$\overline{\phi}(f) = \int_K f\,d\mu \qquad (f \in C(K)).$$

In particular,

$$\phi(e_\lambda) = \int_K e^{\lambda z}\, d\mu(z)$$

for each $\lambda \in \mathbf{C}$. Now the last equation shows that the function Φ defined on the complex plane by

$$\Phi(\lambda) \stackrel{\text{def}}{=} \phi(e_\lambda) \qquad (\lambda \in \mathbf{C})$$

is entire, and—upon differentiating repeatedly under the integral sign—that for each non-negative integer n:

$$\Phi^{(n)}(0) = \int_K \lambda^n \, d\mu = \phi(\text{the monomial function } z \to z^n). \tag{3}$$

But our hypothesis that the functional ϕ annihilates e_λ for each $\lambda \in A$ says that the entire function Φ vanishes at each point of A. Since A has a finite limit point, Φ must vanish on the whole plane, hence the same is true of each of its derivatives. Therefore, by equation (3) the functional ϕ annihilates all monomials, hence all holomorphic polynomials. By Runge's Theorem and the simple connectivity of G, these polynomials are dense in $\mathcal{H}(G)$, hence ϕ is the zero-functional. This completes the proof that $\mathcal{E}(A)$ is dense in $\mathcal{H}(G)$. ///

With these useful criteria for hypercyclicity, non-hypercyclicity, and density in hand, we can proceed immediately to the central result of this paper:

2.8 Theorem (Chaotic characterization of simple connectivity). *Suppose G is a plane domain, and P a nonconstant holomorphic polynomial. Then the following statements are equivalent:*

- *(a) $P(D)$ is hypercyclic on $\mathcal{H}(G)$.*
- *(b) $P(D)$ is chaotic on $\mathcal{H}(G)$.*
- *(c) G is simply connected.*

Proof. (c) $\Rightarrow$ (b): Suppose G is simply connected. We want to show that $P(D)$ is chaotic. This can be inferred, via Runge's Theorem and the continuity of the inclusion map $\mathcal{H}(\mathbf{C}) \to \mathcal{H}(G)$, from the corresponding result for the case $G = \mathbf{C}$, which was proved in [9, Theorem 6.2]. However, in order to illustrate the ideas involved, I will redo the argument of [9] in the context of $\mathcal{H}(G)$.

First we have to show that $P(D)$ is hypercyclic. With an eye towards finding the dense subsets X_0 and Y_0 required by the Hypercyclicity Criterion §2.1, we begin by splitting the plane into two sets. Let U denote the open unit disc, and $\overline{U}$ the closed unit disc, and set

$$A = P^{-1}(U) \qquad \text{and} \qquad B = P^{-1}(\mathbf{C}\backslash\overline{U}).$$

Because P is a polynomial, it has zeros, hence A is not empty. Since P is nonconstant, it is unbounded, hence B is nonempty, and since P is continuous, both A and B are open subsets of the plane. So both A and B have finite limit points, hence the Density Theorem insures that both

$$X_0 \stackrel{\text{def}}{=} \mathcal{E}(A) \qquad \text{and} \qquad Y_0 \stackrel{\text{def}}{=} \mathcal{E}(B)$$

are dense in $\mathcal{H}(G)$.

Now $\lambda \in A$ means that $|P(\lambda)| < 1$, so for n a positive integer,

$$P(D)^n e_\lambda = P(\lambda)^n e_\lambda \to 0 \qquad (n \to \infty),$$

hence $P(D)^n \to 0$ pointwise on X_0.

As for the mapping S, define it on the exponential basis for Y_0 by:

$$S(e_\lambda) = \frac{1}{P(\lambda)} e_\lambda \qquad (\lambda \in B),$$

and extend linearly to Y_0. Since TS is the identity on this exponential basis, it is the identity map on Y_0. Now $\lambda \in B$ means that $|P(\lambda)| > 1$, hence

$$S^n e_\lambda = \frac{1}{P(\lambda)^n} e_\lambda \to 0,$$

so $S^n \to 0$ pointwise on Y_0. Thus all the conditions needed for the Hypercyclicity Criterion are satisfied, and therefore $P(D)$ is hypercyclic.

To prove that $P(D)$ is *chaotic* we need to find a dense set of periodic points. Since the range of P is an unbounded open set that contains the origin, its intersection with the unit circle contains an arc. Let R denote the collection of roots of unity that lie in this arc—they form a dense subset of that arc. Then $C \stackrel{\text{def}}{=} P^{-1}(R)$ is a subset of the plane that has a limit point (it has infinitely many elements, and it is bounded), so by the Density Theorem, the subspace $\mathcal{E}(C)$ is dense in $\mathcal{H}(G)$. Now if $a \in A$ then $P(a)$ is a root of unity, i.e., $P(a)^n = 1$ for some positive integer n. It follows that $P(D)e_a = P(a)^n e_a = e_a$, so e_a is a periodic point of $P(D)$ with period n. Now any linear combination of such periodic vectors is again one (with period equal to the least common multiple of the periods of the component eigenvectors), so every vector in $\mathcal{E}(C)$ is a periodic point of $P(D)$. Thus $\mathcal{E}(C)$ is the desired dense subset of periodic points, hence $P(D)$ is chaotic.

(b) $\Rightarrow$ (a): Trivial from the definition of "chaotic".

(a) $\Rightarrow$ (c): We will prove the contrapositive statement. Suppose G is *not* simply connected. We will show that $P(D)^*$ has an eigenvalue, and therefore by the "non-hypercyclicity criterion" of §2.4 we will know that $P(D)$ is not hypercyclic.

Since G is not simply connected there is a smooth Jordan curve γ in G that surrounds some point $a \notin G$. This curve induces a continuous linear functional ϕ on $\mathcal{H}(G)$ by means of the equation:

$$\phi(f) \stackrel{\text{def}}{=} \int_\gamma f(z)\,dz \qquad (f \in \mathcal{H}(G)).$$

I claim that ϕ is an eigenvector of $P(D)^*$. Note that $\phi \neq 0$, since it takes the value $2\pi i$ at the function $(z-a)^{-1}$ (this is the crucial use of non-simple connectivity). On the other hand, the null space of ϕ contains every derivative: $\phi(Df) = 0$ for every $f \in \mathcal{H}(G)$. Thus, upon writing $P(z) = P(0) + zQ(z)$, where Q is also a holomorphic polynomial, we see that for each $f \in \mathcal{H}(G)$:

$$\begin{aligned}
(P(D)^*\phi)(f) &\stackrel{\text{def}}{=} \phi(P(D)f) \\
&= \phi(P(0)f + DQ(D)f) \\
&= \phi(P(0)f) + \phi(\text{a derivative}) \\
&= P(0)\phi(f),
\end{aligned}$$

that is, $P(D)^*\phi = P(0)\phi$. Thus ϕ is an eigenvector of $P(D)^*$ (for the eigenvalue $P(0)$), and so $P(D)$ is not hypercyclic. ///

2.9 Infinite order differential operators. Suppose $F(z) = \sum_{k=0}^{\infty} a_k z^k$ is a nonconstant entire function for which the series $F(D) = \sum_k a_k D^k$ converges pointwise on $\mathcal{H}(G)$. By the Closed Graph Theorem the linear transformation $F(D)$ so defined is continuous on $\mathcal{H}(G)$, and the proof above works as well for $F(D)$ as it did for $P(D)$. It turns out that the desired convergence always happens if F is an entire function of "exponential type zero" (we will discuss this matter in Section 4), so the above characterization of simple connectivity can be improved by replacing the polynomial P by any nonconstant entire function of exponential type zero.

3 Beyond $P(D)$

Does Theorem 2.8 remain true if the constant coefficient differential operators $P(D)$ are replaced by arbitrary nonscalar operators that commute with D? The answer is "yes and no." I'll begin with the "no" part: There is a non-simply connected domain G for which $\mathcal{H}(G)$ supports a chaotic operator that commutes with D. More precisely:

3.1 Theorem. *Let G be the complex plane with the integers removed. Then the operator T of "translation by one" is chaotic on $\mathcal{H}(G)$.*

Proof. Recall that T is the operator defined by:

$$Tf(z) = f(z+1) \qquad (f \in \mathcal{H}(G) \text{ and } z \in \mathbf{C}).$$

The first order of business is to show that T is hypercyclic, i.e. to find the subspaces X_0 and Y_0 and the inverting operator S required by the Hypercyclicity Criterion (§2.1).

Let

$$\mathcal{F}_+ = \{e_\lambda : \operatorname{Re}\lambda > 0\} \quad \text{and} \quad \mathcal{F}_- = \{e_\lambda : \operatorname{Re}\lambda < 0\},$$

let

$$\mathcal{R} = \left\{ \frac{1}{(z-n)^\alpha} : n \in \mathbf{Z} \text{ and } \alpha \in \mathbf{N} \right\}, \tag{4}$$

and set

$$X_0 = \operatorname{span}\{\mathcal{R} \cup \mathcal{F}_-\} \quad \text{and} \quad Y_0 = \operatorname{span}\{\mathcal{R} \cup \mathcal{F}_+\}.$$

By the Density Theorem (§2.7), each of the sets $\mathcal{F}_+$ and $\mathcal{F}_-$ spans a dense subspace of $\mathcal{H}(\mathbf{C})$, so the closures in $\mathcal{H}(G)$ of both X_0 and Y_0 contain the monomials $\{z^n : n \geq 0\}$. Thus the linear span of the union of $\mathcal{R}$ with these monomials lies in the closures of both X_0 and Y_0, so by Runge's theorem, both X_0 and Y_0 are dense in $\mathcal{H}(G)$.

Clearly $T^n \to 0$ pointwise on $\mathcal{R}$. Since $Te_\lambda = e^\lambda e_\lambda$, the same is true on $\mathcal{F}_-$, and therefore on all of X_0. Let S to be the operator of "translation by -1", i.e. $S = T^{-1}$. Then arguing as above, $S \to 0$ pointwise on Y_0. Thus the requirements of the Hypercyclicity criterion are satisfied, so T is hypercyclic on $\mathcal{H}(G)$.

To show that T is *chaotic* requires a bit more work, since the obvious periodic points e_λ for $\lambda = 2\pi i q$, where q is a (real) rational number, no longer span a dense subspace of $\mathcal{H}(G)$. Fortunately there is another supply of eigenvectors available: I claim that for each point λ of the unit circle and each positive integer α, the series

$$\sum_{n \in \mathbf{Z}} \frac{\lambda^n}{(z-n)^\alpha} \tag{5}$$

converges in $\mathcal{H}(G)$ to an eigenvector $f_{\lambda,\alpha}$ of T corresponding to the eigenvalue λ. Here "convergence" means that the sequence of symmetric partial sums

$$S_N(z) \stackrel{\text{def}}{=} \sum_{n=-N}^{N} \frac{\lambda^n}{(z-n)^\alpha} \tag{6}$$

converges in $\mathcal{H}(G)$.

If $\alpha \geq 2$ then the desired convergence is elementary. For if K is a compact subset of G then, uniformly over K,

$$|z-n|^{-\alpha} = O(|n|^{-\alpha}) \qquad \text{as } |n| \to \infty,$$

hence the "absolute series"

$$\sum_{n \in \mathbf{Z}} \left| \frac{1}{(z-n)^\alpha} \right|$$

converges uniformly on K, because $\alpha > 1$.

However we will also need the case $\alpha = 1$, and this requires more work. If $\lambda = 1$ there is no difficulty: just group corresponding terms of positive and negative index to obtain:

$$S_N(z) = \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2 - n^2},$$

from which the desired convergence is evident.

If $\lambda \neq 1$ then Dedekind's Test [14, Theorem 4, page 137] saves the day. For numerical series this states that $\sum a_n b_n$ converges whenever the partial sums of $\sum a_n$

form a bounded sequence, and the sequence (b_n) converges to zero and has bounded variation, i.e. $\sum |b_n - b_{n+1}| < \infty$. The result is proved by summation-by-parts, and the proof works equally well for two-sided series. Furthermore, the proof shows that if (b_n) is a sequence of *functions* which, on some set S, converges uniformly to zero and has uniformly bounded variation, then the series $\sum a_n b_n$ converges uniformly on S.

Apply this last observation with $a_n = \lambda^n$ and $b_n(z) = (z-n)^{-1}$. Because $|\lambda| = 1$ and $\lambda \neq 1$, the sequence of numerical sums $\sum_{-N}^{N} \lambda^n$ is bounded, so we need only show that the "multiplying sequence" $((z-n)^{-1} : n \in \mathbf{Z})$, which clearly converges to zero in $\mathcal{H}(G)$, also has variation bounded uniformly on each compact subset of G. This, in turn, follows from the fact that the magnitude of the difference between terms of index n and $n+1$ is $|(z-n)(z-n-1)|^{-1}$, which, on each compact subset of G, is uniformly $O(|n|^{-2})$ as $|n| \to \infty$ (the "big-oh constant" depending, of course, on the particular compact set).

Summarizing: for each $\alpha \in \mathbf{N}$ and each complex number λ of modulus one, the series (5) converges in $\mathcal{H}(G)$ to a function $f_{\lambda,\alpha}$, which is an eigenvector of T with eigenvalue λ. Let $\mathcal{F}_\Sigma$ denote the collection of all these eigenvectors where λ is a root of unity, so $\mathcal{F}_\Sigma$ is a set of periodic points of T. Let

$$\mathcal{F}_0 = \{e_\lambda : \lambda \text{ is a root of unity}\},$$

another collection of periodic points for T. Thus the linear span of $\mathcal{F}_0 \cup \mathcal{F}_\Sigma$ also consists entirely of periodic points, and I claim that this subspace is dense in $\mathcal{H}(G)$. This will complete the proof that T is chaotic on $\mathcal{H}(G)$.

Recall that the Density Theorem insures that $\mathcal{F}_0$ spans $\mathcal{H}(\mathbf{C})$, so in particular its closed span picks up all the monomials z^n for n a non-negative integer. So by Runge's Theorem it is enough to show that the closure of the span of $\mathcal{F}_\Sigma$ contains the set $\mathcal{R}$ defined by (4) above.

This is another job for the Hahn-Banach Theorem. Suppose $\phi \in \mathcal{H}(G)^*$ annihilates every function in $\mathcal{F}_\Sigma$. By Hahn-Banach it is enough to prove that ϕ also annihilates every function in $\mathcal{R}$. In plain English, we are assuming that $\phi(f_{\lambda,\alpha}) = 0$ for all $|\lambda| = 1$ and $\alpha \in \mathbf{N}$, and we want to prove that $\phi((z-n)^{-\alpha}) = 0$ for $\alpha \in \mathbf{N}$ and $n \in \mathbf{Z}$.

Fix the positive integer α, and for convenience of notation, let's define the function $r_n \in \mathcal{H}(G)$ by $r_n(z) = (z-n)^{-\alpha}$, and write

$$a_n = \phi(r_n) \qquad (n \in \mathbf{Z})$$

for the quantities that we hope to prove are zero. From the continuity of ϕ and the convergence in $\mathcal{H}(G)$ of the series (5) to $f_{\lambda,\alpha}$ it follows that

$$\sum_{n \in Z} a_n \lambda^n = \phi(f_{\lambda,\alpha}) = 0 \qquad (\text{all } |\lambda| = 1). \tag{7}$$

The desired result now follows from Riemann's Theorem which asserts that if a trigonometric series converges to zero at every point of the unit circle, then the coefficients must all be zero [21, Theorem IX.3.1, page 326]. But it's not necessary to be so fancy: I claim that the left-hand side of (7) has square-summable coefficients, so the more familiar uniqueness theorem from the L^2 theory of Fourier series will also do the job.

For this, note that—as previously observed—the continuity of ϕ means that there is a compact subset K of G and a positive constant C such that

$$|\phi(f)| \leq C||f||_K \qquad (f \in \mathcal{H}(G)).$$

In particular, for each $n \in \mathbf{Z}$:

$$|a_n| = |\phi(r_n)| \leq C\,||r_n||_K = C\,\max_{z\in K}\left|\frac{1}{z-n}\right|^{\alpha} \leq \frac{C'}{(|n|+1)^{\alpha}},$$

where C' is a finite positive number whose actual value depends on C and K (the finiteness of C' arises from the fact that K, being compact, lies a positive distance from $\mathbf{Z}$). Since $\alpha \geq 1$ this proves that $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$, which puts the right-hand side of (7) into L^2 of the unit circle. Thus by (7) all the coefficients a_n must be zero. This completes the proof that T is chaotic. ///

It is no accident that the domain G of the last theorem is infinitely connected. The next result—the "yes" answer to the question that led off this section—shows that for *finitely connected domains* any nonscalar operator in the commutant of D can be used to characterize simple connectivity.

3.2 Theorem. *Suppose G is a finitely connected domain and L a non-scalar operator on $\mathcal{H}(G)$ that commutes with D. Then the following are equivalent:*

(a) L is hypercyclic on $\mathcal{H}(G)$.

(b) L is chaotic on $\mathcal{H}(G)$.

(c) G is simply connected.

PROOF. The goal is to show that L is chaotic when G is simply connected, and that it fails to be hypercyclic when G is not simply connected.

So first suppose that G is simply connected. The idea behind proving L chaotic is to represent L in terms of D and try to adjust the proof of Theorem 2.8 so that it works in the new situation. Since $LD = DL$ on $\mathcal{H}(G)$, for each $\lambda \in \mathbf{C}$ the equation $De_\lambda = \lambda e_\lambda$ implies

$$DLe_\lambda = LDe_\lambda = L(\lambda e_\lambda) = \lambda Le_\lambda,$$

i.e., the function $y = Le_\lambda$ satisfies the complex differential equation $y' = \lambda y$ on $\mathbf{C}$. Thus for each $\lambda \in \mathbf{C}$ there is a complex number $F(\lambda)$ such that

$$Le_\lambda = F(\lambda)e_\lambda. \tag{8}$$

I claim that F is an entire function. Indeed, fix a point $z_0 \in G$, and note that for each fixed $\lambda \in \mathbf{C}$ the right-hand side of the series representation

$$e_\lambda(z) \stackrel{\text{def}}{=} e^{\lambda z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \lambda^n$$

converges in $\mathcal{H}(G)$ to e_λ. By the definition of F and the continuity of both the operator L and the linear functional of evaluation at z_0,

$$F(\lambda) e^{\lambda z_0} = (Le_\lambda)(z_0) = \sum_{n=0}^{\infty} \frac{(L(z^n))(z_0)}{n!} \lambda^n \ ,$$

where on the right you see a convergent numerical series. Thus F is represented by a power series that converges at each point of the plane, hence it is an entire function.

Since G is simply connected we know from the Density Theorem (§2.7) that the complex exponential functions span a dense subspace of $\mathcal{H}(G)$. Now in Theorem 2.8 the proof that $P(D)$ is chaotic depended only on the action of that operator on the linear span of the complex exponentials. Once we know (8) and that F is entire, the same proof works word for word, with F in place of P, to prove that L is chaotic.

Now suppose that G is *not* simply connected. We are supposed to show that L is not hypercyclic. Since G is finitely connected there is a positive integer N such that the complement of G has exactly N components $K_1, \ldots, K_N$, where K_N is the unbounded component. Let $\gamma_1, \ldots, \gamma_N$ be disjoint rectifiable Jordan curves in G, with γ_j surrounding K_j but no other K_i, and γ_N surrounding all the other γ_j's. In other words, let $\gamma_1, \ldots, \gamma_N$ be a *homology basis* for G in the sense of Ahlfors's book [1, page 146] (for Rudin's version of this see [18, Theorem 13.5, page 268]).

Associated with each curve γ_j there is a linear functional ϕ_j on $\mathcal{H}(G)$ defined by:

$$\phi_j(f) = \int_{\gamma_j} f(z)\, dz \qquad (f \in \mathcal{H}(G)).$$

As in the proof that (a) $\Rightarrow$ (c) of Theorem 2.8, the functional ϕ_j annihilates every derivative in $\mathcal{H}(G)$. Moreover it follows from Cauchy's theorem that every member of $\mathcal{H}(G)$ that is annihilated by each the functionals ϕ_j for $1 \le j \le N-1$ is a derivative (see [1, page 146]). Let $\operatorname{ran} D$ denote the range of the operator D, i.e. the subspace of $\mathcal{H}(G)$ consisting of derivatives. The previous remarks say that the annihilator of $\operatorname{ran} D$, defined by

$$(\operatorname{ran} D)^\perp \stackrel{\text{def}}{=} \{\phi \in \mathcal{H}(G)^* : \phi(f) = 0 \text{ for every } f \in \operatorname{ran} D\},$$

has $\{\phi_1, \ldots, \phi_{N-1}\}$ as a basis (the functionals are clearly linearly independent, and by Cauchy's theorem ϕ_N is a linear combination of $\phi_1 \ldots \phi_{N-1}$)), and so has dimension $N - 1 > 0$.

But L commutes on $\mathcal{H}(G)$ with the operator D, so $L(\operatorname{ran} D) \subset \operatorname{ran} D$. From this it follows easily (just as for Hilbert or Banach space operators) that $L^*((\operatorname{ran} D)^\perp) \subset (\operatorname{ran} D)^\perp$. Thus L^* has a nontrivial finite dimensional invariant subspace, so it has an eigenvalue, and therefore L satisfies the "non-hypercyclicity criterion" (Theorem 2.4). ///

3.3 Open Question. I don't know if, for *every* infinitely connected domain G, there is a chaotic (or even just a hypercyclic) operator on $\mathcal{H}(G)$ that commutes with differentiation.

4 Operators that commute with differentiation

Lurking behind the work of Sections 2 and 3 is the problem of finding concrete representations for operators that commute with differentiation. I take this up in more detail now. I'm sure most, if not all, of the results of this section are already known, so what follows should be treated as mostly expository.

4.1 Notation. For a plane domain G, not necessarily simply connected, $\mathrm{Com}_D(G)$ will denote the collection of operators that commute with D on $\mathcal{H}(G)$—the *commutant* of D on $\mathcal{H}(G)$.

4.2 The characteristic function. The fundamental tool for studying the commutant already resides in the proof of Theorem 3.2, wherein it developed that each $L \in \mathrm{Com}_D(G)$ gives rise to an entire function F defined by the equation

$$Le_\lambda = F(\lambda)e_\lambda \qquad (\lambda \in \mathbf{C}), \tag{9}$$

which, you will recall, comes from the fact that the function $y = Le_\lambda$ satisfies the differential equation $y' = \lambda y$. In order to emphasize the connection between F and L, in this section we'll write $F = F_L$, and refer to it as the "characteristic function" of L.

4.3 Entire functions of differentiation. Let us say that an entire function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *G-operates* on D if the operator series $\sum_{n=0}^{\infty} a_n D^n$ converges pointwise on $H(G)$. In this case we will write

$$F(D) = \sum_{n=0}^{\infty} a_n D^n \tag{10}$$

When this happens the Closed Graph Theorem insures that $F(D)$, which is clearly a linear mapping on $\mathcal{H}(G)$, is also continuous, and you can check easily that the characteristic function of $F(D)$ is just F.

Two classes of entire functions play a pivotal role in what is to follow:

4.4 Definition. An entire function F is said to be of *exponential type* if there exist positive constants A and B such that

$$|F(z)| \le A\, e^{B|z|} \qquad (\text{all } z \in \mathbf{C}). \tag{11}$$

If, in addition, given *any* positive B there is a constant A such that (11) holds, then F is said to be of *exponential type zero.*

The next two results show the importance of the notion of "exponential type."

4.5 Lemma. *If $L \in Com_D(G)$, then its characteristic function F_L is of exponential type.*

Proof. Recall the basic neighborhood of zero $N(K,\varepsilon)$ that we encountered in the proof of Theorem 2.7. Since L is continuous, for every compact subset K of G and $\varepsilon > 0$, the set $L^{-1}(N(K,\varepsilon))$ is also a neighborhood of zero. Thus there exists another compact subset J, which we may assume contains K, and a positive number δ such that $L(N(J,\delta)) \subset N(K,\varepsilon)$. Take $\varepsilon = 1$ and set $A = 1/\delta$. Then

$$||Lf||_K \le A||f||_J \qquad (\text{all } f \in \mathcal{H}(G)). \tag{12}$$

We may without loss of generality suppose that G contains the origin. Then inequality (12), with K taken to be the singleton containing the origin, asserts that

$$|F(\lambda)| = |F(\lambda))e_\lambda(0)| = |Le_\lambda(0)| \le A||e_\lambda||_J \le Ae^{B|\lambda|},$$

where $B = \max\{|z| : z \in G\}$. ///

The issue now is whether or not every entire function of exponential type G-operates on D. The answer depends on the nature of G. First, the good news:

4.6 Theorem. *Every entire function of exponential type* $\mathbf{C}$*-operates on D.*

Proof (cf. [9, §5.3], for example). Fix $F(z) = \sum_{n=0}^{\infty} a_n z^n$, an entire function of exponential type. Our goal is to show that the series $\sum_{n=0}^{\infty} a_n D^n$ converges pointwise on $\mathcal{H}(G)$. The restriction on F is that there exist positive constants A and B such that F obeys (11) for all $z \in \mathbf{C}$. An elementary orthogonality argument shows that for every $r > 0$:

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})e^{-in\theta}\,d\theta,$$

which, in view of the growth restriction on F, implies that

$$|a_n| \le Ae^{Br}r^{-n} = AB^n e^{Br}(Br)^{-n} \qquad (\text{all } r > 0).$$

In particular, the choice $r = n/B$ (which, incidentally, minimizes the right-hand side of the last equation), yields the crucial inequality:

$$|a_n| \le A\left(\frac{Be}{n}\right)^n \qquad (n = 1, 2, \ldots). \tag{13}$$

The next order of business is to obtain an estimate on derivatives of entire functions. Suppose $f \in \mathcal{H}(\mathbf{C})$, fix a compact subset K of $\mathbf{C}$, and set $\rho = \max\{|z| : z \in K\}$. Let $r = \rho + 2Be$, so, in particular, K lies in the disc $\{|z| < r\}$. Finally, set $M = \max\{|f(\zeta)| : \zeta = r\}$.

Then the Cauchy formula for derivatives provides the following estimate, valid for every non-negative integer n and every $z \in K$:

$$\begin{aligned} |D^n f(z)| &= \frac{n!}{2\pi}\left|\int_{|\zeta|=r} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta\right| \\ &\leq \frac{n!\, r\, M}{(r-\rho)^{n+1}} \\ &= C\,\frac{n!}{(2Be)^n}, \end{aligned}$$

where $C = Mr/(r-\rho)$, a constant independent of $n \in \mathbf{N}$ and $z \in K$.

This inequality and (13) yield, for each $z \in K$ and $n \in \mathbf{N}$:

$$|a_n||D^n f(z)| \leq AC\left(\frac{Be}{n}\right)^n \frac{n!}{(2Be)^n} = AC\frac{n!}{n^n}\frac{AC}{2^n} \leq \frac{AC}{2^n},$$

which shows that the series $\sum_{n=0}^{\infty} a_n D^n f(z)$ converges uniformly on K. Since K is an arbitrary compact subset of G, the series $\sum_{n=0}^{\infty} a_n D^n f$ does indeed converge in $\mathcal{H}(\mathbf{C})$, as we wished to show. ///

4.7 Corollary. $\mathrm{Com}_D(\mathbf{C}) = \{F(D) : F \text{ is entire, of exponential type}\}$.

PROOF. By the previous theorem, if F is entire of exponential type, then F **C**-operates on D, hence the resulting operator clearly $F(D)$ commutes with D. Conversely if L is an operator that commutes with D, then its characteristic function F_L is of exponential type by Lemma 4.5, so the operator $F_L(D)$ exists. It is easy to check that for each $\lambda \in \mathbf{C}$, $F_L(D)e_\lambda = F(\lambda)e_\lambda$, and this, along with (9), guarantees that $L = F_L(D)$ on $\mathcal{E}(\mathbf{C})$, the linear span of the exponential functions e_λ for $\lambda \in \mathbf{C}$. Since $\mathcal{E}(\mathbf{C})$ is dense in $\mathcal{H}(\mathbf{C})$ (Theorem 2.7), it follows that $L = F_L(D)$ on $\mathcal{H}(\mathbf{C})$, as desired. ///

In particular, the translation operator T_a defined in Section 1 by equation (1) can be represented as an entire function of D. I leave to you to prove the following:

4.8 Corollary. *On $\mathcal{H}(\mathbf{C})$, $T_a = e^{aD}$ for each $a \in \mathbf{C}$.*

Here's an example showing that Theorem 4.6 doesn't hold if **C** is replaced by an arbitrary simply connected domain G.

4.9 Example. Let G be the domain formed by gluing to the top edge of a horizontal strip squares of unit length, each one unit apart. So you get a "sawtooth" domain that is invariant under the map $z \to z+1$, and therefore the translation operator T_1 acts (continuously) on $\mathcal{H}(G)$. I claim, however, that the characteristic function e^z of

T_1 does not G-operate D, i.e. that the series $\sum_{n=0}^{\infty} D^n/n!$ does not converge pointwise on $\mathcal{H}(G)$.

Indeed, suppose for the sake of contradiction that the series *does* converge. Then its individual terms converge to zero in $\mathcal{H}(G)$, from which it follows easily that the series $\sum_{n=0}^{\infty} D^n/n!2^n$ converges to some operator L on $\mathcal{H}(G)$. But on the linear span of the exponential functions, L is easily seen to coincide with $T_{1/2}$. Since G is simply connected, this linear span is dense in G (Theorem 2.7), hence L is the restriction to $\mathcal{H}(G)$ of $T_{1/2}$, which does not take $\mathcal{H}(G)$ into itself. This contradiction shows that the original series could not have converged. ///

The notion of *exponential type zero* emerges when we try to determine which entire functions act on D for $G \neq \mathbf{C}$.

4.10 Theorem. *Suppose G is any bounded domain, not necessarily simply connected. Then every function of exponential type zero G-operates on D.*

Proof. Suppose F is an entire function of exponential type zero. To show it G-operates on D we need to redo the Cauchy integral estimate that occurred in the proof of Theorem 4.6. For this, fix a function $f \in \mathcal{H}(G)$, a compact subset K of G, and a cycle γ in $G \backslash K$ that "surrounds" K in the sense that the Cauchy integral formula holds over γ for every $f \in \mathcal{H}(G)$ and every $z \in K$ (see [18, Theorem 13.5, page 268]). Let $d = \operatorname{dist}(\gamma, K)$ and let ℓ denote the length of γ. Then the Cauchy formula for the n-th derivative supplies this estimate, valid for every $z \in K$ and every non-negative integer n;

$$\begin{aligned} |D^n f(z)| &= \frac{n!}{2\pi} \left| \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \right| \\ &\leq \left(\frac{||f||_\gamma}{2\pi \ell d} \right) \frac{n!}{d^n}, \end{aligned}$$

so that

$$||D^n f||_K \leq C ||f||_\gamma \frac{n!}{d^n} \qquad (n = 0, 1, 2, \ldots) \tag{14}$$

where $C = (2\pi \ell d)^{-1}$ is a constant independent of n and f.

Now since F is of exponential type zero, there is a constant $A > 0$ such that F satisfies estimate (11) with $B = d/2e$. From this, the coefficient estimate (13) yields

$$|a_n| \leq A \left(\frac{d}{2n} \right)^n \qquad (n = 1, 2, \ldots).$$

Therefore

$$|a_n| \, ||D^n f||_K \;\; \leq \;\; AC \, ||f||_\gamma \frac{n!}{d^n} \left(\frac{d}{2n} \right)^n$$

$$= AC\,||f||_\gamma \left(\frac{1}{2}\right)^n \frac{n!}{n^n}$$

$$\leq AC\,||f||_\gamma \left(\frac{1}{2}\right)^n .$$

This shows that the series $\sum_n a_n D^n f$ converges uniformly on K, and since K is an arbitrary compact subset of G and f an arbitrary function in $\mathcal{H}(G)$, it follows that the operator series $\sum_n a_n D^n$ converges pointwise on $\mathcal{H}(G)$ (note also that our estimate explicitly reveals the continuity of the operator represented by the sum). Thus the entire function, $F(z) = \sum_n a_n z^n$, G-operates on D. ///

Now that Theorem 4.10 is proved, the following consequence, noted in §2.9, is official:

4.11 Corollary. *Theorem 2.8 remains true if the phrase "P is a nonconstant holomorphic polynomial" is replaced by "P is a nonconstant entire function of exponential type zero."*

The proof of Theorem 4.10 actually provides more than was promised. Suppose F is entire, of exponential type zero, and set $L = F(D)$. Suppose the compact subset K of G is given. For any $0 < \varepsilon < \operatorname{dist}(K, \partial G)$ let K_ε denote the "ε-envelope" of K: those points of G that lie within ε of some point of K. The restriction on ε insures that K_ε lies entirely in G. Now the cycle γ in the proof of Theorem 4.10 can be chosen to lie in K_ε, from which the original argument then yields a constant $A = A(K, \varepsilon)$ such that

(15) $$||Lf||_K \leq A\,||f||_{K_\varepsilon} \qquad (\text{all } f \in \mathcal{H}(G)).$$

By contrast, recall that L is continuous precisely when, given any compact $K \subset G$, there is a constant A and a compact subset J of G (which may, without loss of generality, be assumed to contain K), such that $||Lf||_K \leq A\,||f||_J$ for each $f \in \mathcal{H}(G)$. Thus inequality (15) expresses a property of L that is stronger than continuity, and which I'd like to formalize in the following definition.

4.12 Definition. A linear transformation L on $\mathcal{H}(G)$ is *supercontinuous* if: for every $0 < \varepsilon < \operatorname{dist}(K, \partial G)$, and every compact $K \subset G$, there is a constant $A = A(K, \varepsilon) > 0$ such that inequality (15) holds.

Thus the the proof of Theorem 4.10 actually provides this:

> *For any plane domain G, every entire function F of exponential type zero G-operates on D, and the resulting operator $F(D)$ is* supercontinuous *on $\mathcal{H}(G)$.*

For an example of an operator that is *not* supercontinuous, consider T_a, the operator of translation by a nonzero complex number a, acting on $\mathcal{H}(\mathbf{C})$. It is easy to check that T_a is not supercontinuous on $\mathcal{H}(\mathbf{C})$. Note that by Theorem 4.6 and Corollary 4.8, the exponential function e_a $\mathbf{C}$-operates on $\mathcal{H}(\mathbf{C})$, and $T_a = e_a(D)$. This motivates the next result, which shows that, for *simply connected* domains, supercontinuity for an operator in the commutant of D is *equivalent* to exponential type zero for its characteristic function. The result also shows that, for such operators, supercontinuity is equivalent to two apparently weaker properties that involve the linear functionals of evaluation at points of G.

4.13 Definition. For $z \in G$ let ϕ_z denote the "evaluation functional" defined on $\mathcal{H}(G)$ by:

$$\phi_z(f) = f(z) \qquad (f \in \mathcal{H}(G)).$$

It's an easy exercise to show that each of these evaluation functionals is supercontinuous on $\mathcal{H}(G)$.

4.14 Theorem. *Suppose G is simply connected and $L \in \mathrm{Com}_D(G)$. Then following statements about L are equivalent:*

(a) The characteristic function F of L is of exponential type zero (so $L = F(D)$).

(b) L is supercontinuous.

(c) $L^*\phi_z$ is supercontinuous on $\mathcal{H}(G)$ for every $z \in G$.

(d) $L^*\phi_z$ is supercontinuous on $\mathcal{H}(G)$ for some $z \in G$.

Proof. The implication (a) $\Rightarrow$ (b) was noted above.

(b) $\Rightarrow$ (c): Since $L * \phi_z = \phi_z \circ L$, this follows from the fact that a composition of supercontinuous functions is supercontinuous. In the special case being considered here, the supercontinuity asserted for $L^*\phi_z$ translates into the assertion that for every $0 < \varepsilon < \mathrm{dist}\,(z, \partial G)$ there exists a constant $A > 0$ such that

$$|Lf(z)| \leq A\,||f||_{B(\varepsilon,z)} \qquad (f \in \mathcal{H}(G)), \tag{16}$$

where $B(\varepsilon, z)$ is the closed ball of radius ε centered at z. But (16) is just what you get by taking K to be the singleton containing z in the definition of supercontinuity for L.

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a): The hypothesis is that $L^*\phi_z = \phi_z \circ L$ is supercontinuous at a point $z \in G$, and the goal is to prove that the characteristic function F of L is of exponential type zero. Recall that "F is the characteristic function of L" means that $Le_\lambda = F(\lambda)e_\lambda$

for each $\lambda \in \mathbf{C}$. Fix $0 < \varepsilon < \operatorname{dist}(z, \partial G)$ and $\lambda \in \mathbf{C}$. Then the hypothesis on $L^*\phi_z$ provides a positive constant $A = A(\varepsilon, z) > 0$ such that:

$$\begin{aligned}
|F(\lambda)| \exp(\operatorname{Re}(\lambda z)) &= |F(\lambda) e_\lambda(z)| \\
&= |(Le_\lambda)(z)| \\
&= |\phi_z(Le_\lambda)| \\
&\stackrel{\text{def}}{=} |L^*(\phi_z)(e_\lambda)| \\
&\leq A\, ||e_\lambda||_{B(z,\varepsilon)} \\
&= A \exp\left(\max_{|\zeta - z| \leq \varepsilon} \operatorname{Re}(\lambda\zeta) \right).
\end{aligned}$$

Thus

$$|F(\lambda)| \leq A \exp\left(\max_{|\zeta - z| \leq \varepsilon} \operatorname{Re}[\lambda(\zeta - z)] \right) \leq A\, e^{\varepsilon|\lambda|}$$

Since λ is any point of the complex plane, and the constant A does not depend on λ, this proves that F is of exponential type zero. ///

The next result uses the idea of the previous proof to show that supercontinuity for L and exponential type zero for F_L are equivalent for *bounded* simply connected domains.

4.15 Theorem. *If G is a bounded, simply connected domain, and F an entire function that G-operates on D, then F is of exponential type zero.*

PROOF. Choose an increasing sequence $\{G_n\}$ of open subsets of G whose union is G, and such that the closure of G_n (compact because G is bounded) lies in G_{n+1}. For $f \in \mathcal{H}(G)$ let $||f||_n = \sup\{|f(z)| : z \in G_n\}$.

Let $L = F(D)$, so that L is continuous on $\mathcal{H}(G)$ and $Le_\lambda = F(\lambda)e_\lambda$. The continuity of L implies that for each n there is an index $m \geq n$ and a constant $A_n > 0$ such that

$$||Lf||_n \leq A_n ||f||_m \qquad (\text{all } f \in \mathcal{H}(G)). \tag{17}$$

Fix a complex number ω of modulus one, and a positive real number r. Then—as in the previous proof—for every $z \in G_n$,

$$\begin{aligned}
|F(r\omega)| \exp(r \operatorname{Re}(\omega z)) &= |F(r\omega) e^{r\omega z}| \\
&= |L(e_{r\omega}(z)| \\
&\leq ||L(e_{r\omega})||_n
\end{aligned}$$

$$\begin{aligned} &\leq A_n \, ||e_{r\omega}||_m \\ &= A_n \sup_{z \in G_m} \exp(\mathrm{Re}\,(r\omega z)). \end{aligned}$$

Now let $p(\omega) = \sup_{z\in G} \mathrm{Re}\,(\omega z)$, and write $p_n(\omega)$ for the corresponding supremum over G_n. Then the last chain of inequalities yields:

$$|F(r\omega)| \exp(r\mathrm{Re}\,(\omega z)) \leq A_n \exp(r\, p_m(\omega)) \leq A_n \exp(r\, p(\omega)),$$

from which it follows, upon taking the supremum of the left-hand side over $z \in G_n$, that

$$|F(r\omega)| \exp(r\, p_n(\omega)) \leq A_n \, \exp(r\, p(\omega)),$$

Summarizing: for each index n, each $r > 0$ and each ω in the unit circle:

$$|F(r\omega)| \leq A_n \, \exp(r[p(\omega) - p_n(\omega)]). \tag{18}$$

Now the functions p_n and p are continuous on the unit circle—it is an elementary exercise to show that for each pair of points ω_1, ω_2 on the unit circle,

$$|p(\omega_1) - p(\omega_2)| \leq \left(\max_{z\in G} |z|\right) |\omega_1 - \omega_2|$$

(I thank Paul Bourdon for pointing this out to me). Clearly the sequence of functions $\{p_n\}$ increases pointwise to p, so by Dini's theorem, $p_n \to p$ uniformly on the unit circle.

Suppose $\varepsilon > 0$ is given. Use the uniform convergence mentioned above to choose n so that $p(\omega) - p_n(\omega) < \varepsilon$ for each ω in the unit circle. Then (18) shows that for each $r > 0$ and ω in the unit circle,

$$|F(r\omega)| \leq A_n \, e^{r\varepsilon},$$

in other words: F is of exponential type zero. ///

4.16 Corollary. *If G is a bounded, simply connected domain, then $\mathrm{Com}_D(G)$ consists of all operators $F(D)$, where F is an entire function of exponential type zero.*

Proof. The proof given above actually showed that if L commutes with D then the characteristic function F_L is of exponential type zero, so in particular, $L = F_L(D)$. The rest follows as in the proof of Corollary 4.7. ///

4.17 Remark. The function p that occurred in the the proof of Theorem 4.15 occurs frequently in convex analysis and in the theory of entire functions: it is called the *support function* of the complex conjugate of G.

5 Simple connectivity and cyclicity

This final section investigates the possibility of replacing "hypercyclic" with "cyclic" in Theorem 2.8. In other words:

> *If G is a plane domain that is* not *simply connected, does $P(D)$ fail to be cyclic on $\mathcal{H}(G)$ for every nonconstant holomorphic polynomial P?*

Now if D itself fails to be cyclic, then the same will be true of $P(D)$ for each polynomial P, so the discussion will focus on the differentiation operator itself. As you'll see next, the methods of Section 2 quickly rule out domains of connectivity larger than two.

5.1 Proposition. *If G has connectivity larger than two, then D is not cyclic on $\mathcal{H}(G)$.*

Proof. The assumption on G is that its complement has (at least) two distinct bounded components K_1 and K_2. Let γ_1 be a simple, closed, rectifiable curve in G that surrounds K_1 but not K_2, and similarly let γ_2 surround K_2 but not K_1. Then for $j = 1, 2$ define the linear functional ϕ_j on $\mathcal{H}(G)$ by:

$$\phi_j(f) = \int_{\gamma_j} f(z)\,dz \qquad (f \in \mathcal{H}(G)).$$

As in the proof of Theorem 2.1, neither of these is the zero-functional, and both annihilate all derivatives. In other words, the range of the differentiation operator lies in the intersection of the null spaces of these two functionals. Now the functionals themselves are linearly independent (for example, if $z_1 \in K_1$ and $f(z) = 1/(z - z_1)$, then $\phi_1(f) \neq 0$, but $\phi_2(f) = 0$, so ϕ_1 and ϕ_2 cannot be constant multiples of each other), so this intersection has codimension at least two. Thus the same is true of the closure of the range of D. But it is easy to see that the closure of the range of a *cyclic operator* can have codimension no more than one. Thus D is not cyclic on $\mathcal{H}(G)$. ///

5.2 Remark. We have previously observed (§2.4) that if T is an operator on a Fréchet space, and T^* has an eigenvalue, then T is not hypercyclic. The argument above shows that if T^* has an eigenvalue of multiplicity larger than 1, then T is not even cyclic.

The case of connectivity two is more difficult, and for this I do not have a complete result. However the following partial result, which requires that G have some additional symmetry about some point of its complement, provides evidence for non-cyclicity in this case, too. It will become clear from the proof that the symmetry hypothesis given here can be somewhat relaxed.

5.3 Theorem. *Suppose G is a doubly connected domain, and let K denote the bounded component of $\mathbf{C}\backslash G$. Suppose further that G contains an annulus that surrounds K. Then D is not cyclic on $\mathcal{H}(G)$.*

Proof. Let Ω denote the annulus. By Runge's Theorem, the restrictions of functions in $\mathcal{H}(G)$ to $\mathcal{H}(\Omega)$ are dense in $\mathcal{H}(\Omega)$, so $\mathcal{H}(G)$ can be regarded as a dense subspace of $\mathcal{H}(\Omega)$. Thus if D were cyclic on $\mathcal{H}(G)$ then it would also be cyclic on $\mathcal{H}(\Omega)$. So it is enough to prove that D is not cyclic on $\mathcal{H}(\Omega)$.

The cyclicity or non-cyclicity of D is not altered by translation, so we may without loss of generality assume that Ω has its center at the origin, say $\Omega = \{z \in \mathbf{C} : r < |z| < R\}$. Let $\Omega_i = \{|z| < R\}$ and $\Omega_o = \{|z| > r\}$.

By Laurent series, each analytic function h on Ω can be written uniquely as $f + g$ where f is analytic in Ω_i and g is analytic in Ω_o and vanishes at ∞. In other words, $\mathcal{H}(G)$ decomposes into a direct sum of closed subspaces $\mathcal{H}_i$ and $\mathcal{H}_o$, where $\mathcal{H}_i = \mathcal{H}(\Omega_i)$ and $\mathcal{H}_o$ consists of those functions in $\mathcal{H}(\Omega_o)$ that vanish at ∞.

Now each subspace $\mathcal{H}_i$ and $\mathcal{H}_o$ is invariant under differentiation, so D, acting on $\mathcal{H}(G)$, decomposes into a direct sum $D \oplus D$ acting on $\mathcal{H}_i \oplus \mathcal{H}_o$. Fix a pair (f, g) in $\mathcal{H}_i \oplus \mathcal{H}_o$. We will show that (f, g) is not cyclic for $D \oplus D$. If f is the zero-function this is trivial since then the orbit of (f, g) lies in $\{0\} \oplus \mathcal{H}_o$, so its span has no chance of being dense in $\mathcal{H}_i \oplus \mathcal{H}_o$. So we may as well suppose that f is not the zero-function.

The rest of the argument uses an unpublished idea due to James Deddens: we will find a nontrivial continuous linear functional on $\mathcal{H}_i \oplus \mathcal{H}_o$ that annihilates the entire $D \oplus D$ orbit of (f, g), hence this orbit cannot have dense linear span in $\mathcal{H}_i \oplus \mathcal{H}_o$.

Fix a circle $\gamma = \{|z| = \rho\}$ where $r < \rho < R$, and for $(F, G) \in \mathcal{H}_i \oplus \mathcal{H}_o$ define

$$\Lambda(F, G) = \int_\gamma [g(-z)F(z) + f(-z)G(z)]\, dz \ .$$

The first order of business is to show that Λ is not the zero-functional. Since f is not the zero-function, one of its derivatives, say the N-th one, does not vanish at the origin. Let $G(z) = z^{-(N+1)}$, a function in $\mathcal{H}_o$. Then

$$\Lambda(0, G) = \frac{1}{n!}\int_\gamma \frac{f(-z)}{z^{N+1}}\, dz = \frac{(-1)^N}{2\pi i} f^{(N)}(0) \neq 0,$$

so Λ is not trivial.

Now fix a non-negative integer n. I claim that $\Lambda(D^n f, D^n g) = 0$, which will complete our proof. It follows upon integrating by parts n times that:

$$\begin{aligned}
\Lambda(D^n f, D^n g) &\stackrel{\text{def}}{=} \int_\gamma [g(-z)D^n f(z) + f(-z)D^n g(z)]\, dz \\
&= \int_\gamma [f(z)D^n g(-z) + f(-z)D^n g(z)]\, dz \\
&= \int_\gamma [H(z) + H(-z)]\, dz \ ,
\end{aligned}$$

where $H(z) = f(z)D^n g(-z)$ for $z \in \Omega$. Parameterize γ, say by $\gamma(t) = \rho \exp(2\pi i t)$ for $0 \le t \le 1$, to obtain:

$$\begin{aligned} \int_\gamma H(-z)\,dz &= \int_0^1 H(-\gamma(t))\gamma'(t)\,dt \\ &= -\int_0^1 H(-\gamma(t))(-\gamma)'(t)\,dt \\ &= -\int_\gamma H(z)\,dz\ , \end{aligned}$$

where the last line follows from the fact that the parameterization $t \to -\rho\exp(2\pi i t)$ (with $0 \le t \le 1$) also represents γ, *with the same orientation.* Thus $\Lambda(D^n f, D^n g) = 0$, as promised. ///

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