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A Stairway to Chaos

by

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Abstract
Dynamical systems theory and system dynamics diverged at some point in the recent past. Here we take a first step toward convergence. This is a concise, visual introduction to the basic concepts of the new theories of chaos and bifurcation.

Contents

1. A history of the divergence

Systems science evolved partly in reaction to destructive aspects of scientific rationalism: reduction and specialization. Along with cybernetics, holism, and related paradigm struggles, a movement was planned with a mathematical core. In fact, in an essay entitled "The History and Development of General Systems Theory" written near the end of his life, Ludwig von Bertalanffy, the founder of General Systems Theory, described his theory retrospectively in three main trends:

- Systems science: mathematical system theory
- Systems technology
- Systems philosophy

These are his names, verbatim, and to them he devoted, respectively, 7, 1, and 5 pages. In any case, we may take as our subject here, *systems science,* also known as *mathematical system theory,* a branch of mathematics.

Within the 7 pages, then, von Bertalanffy described the branch of mathematics now generally known in mathematical circles as *dynamical systems theory.* But as it evolved within the ambiance of general systems theory, it attained the name *system dynamics*. By now this is a specialty with a holistic flavor, and is extensively taught in engineering schools, where it is presented as a technology unifying mechanics, chemistry, heat balance, and so on, in the context of the modeling and simulation of engineering systems.

In fact, due to the hermeneutical circle so important to the history and development of the sciences, in which experiments and models alternate in a spiral of evolution, this modeling context is very important. However, the system dynamics curriculum has not very flexibly followed the new developments of dynamical systems theory, and perhaps this divergence could be remedied. With the advent of the computer revolution and the new mathematics of *complex dynamical systems* (including the theories of chaos, bifurcations, catastrophes, and fractal boundaries), a restorative convergence may now be underway.

In this paper we will review the basic concepts of the new math, and then return to the vision of von Bertalanffy.

2. The stairway to chaos

Dynamics is a vast area, and our subject is a relatively new frontier within it. So, for those who already have an idea of the territory of dynamical systems, we would like now to locate our subject within this larger territory.

a. Four kinds of dynamics.

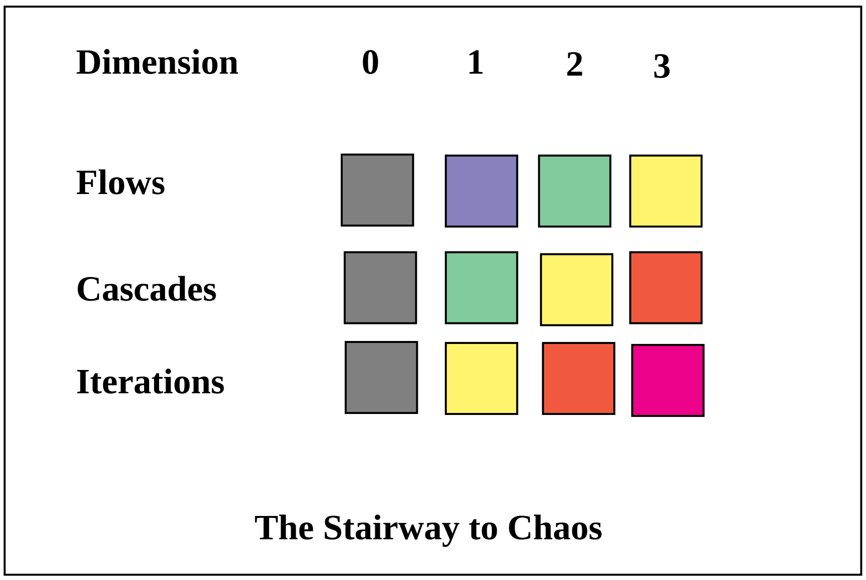
Dynamical systems theory has four flavors:

- *flows* are continuous families of invertible maps generated by a system of autonomous first-order ordinary differential equations, or vectorfield, and parameterized continuously by time, that is, by real numbers;
- *cascades* are discrete families of invertible maps generated by the iteration of a given invertible map and its inverse, and parameterized discretely by the integers (zero, positive, and negative);

- *semi-cascades* are discrete families of maps generated by iteration of a given map, generally noninvertible, and parameterized discretely by the natural numbers (zero and the positive integers).
- *shifts*, or *symbolic dynamical systems*, are systems with discrete time and discrete state spaces.

Both cascades and semi-cascades are also known as *discrete dynamical systems*, or *iterations*. In general, the *state space*, the space in which a flow, cascade, or semi-cascade is defined, may be an arbitrary space of any dimension: 1, 2, 3, and so on. The state space of a shift has dimension 0. This suggests a tableau of types of dynamical systems, as shown in Fig. 0.

.*Figure* 0. The stairway to chaos.



The Stairway to Chaos

In this tableau, there is a relationship between cells on the same diagonal. For example, in each row, the cell marked A is the cell of lowest dimension in which *chaos* occurs. Hence, the tableau is called the *stairway to chaos*. Here chaos might mean any dynamic behavior more complicated than periodic behavior, but in the literature of chaos theory, the word chaos usually is given a more restrictive meaning. The second diagonal, marked B in Fig. 0, may be regarded as the current frontier of chaos theory research.

b. The basic concepts

The main features of a *dynamical system* — flow, cascade, or semicascade — are its *attractors*, their *basins*, and the boundaries of its basins, or *separatrices*. And for a *dynamical scheme*, which means a family of dynamical systems parametrized by so-called control variables, the main features are its *bifurcations*. We are going to introduce these basic concepts now by example, step-by-step on the stairway to chaos.

3. Some examples

Let us label the rows of the tableau of Fig. 0 by F (for flows) for the top row, then C (for cascades) for the middle row, and finally S (for semi-cascades) for the bottom row. We also refer to the columns by numerals indicating the dimension of the state space. Thus the upper left cell is denoted F1, to its right F2, directly below that, C2, and so on.

Figure 1. Cell F1: flow, one-dimensional. This example shows two attractors, each centrally located in its basin, and a single point separatrix. It is a repellor.

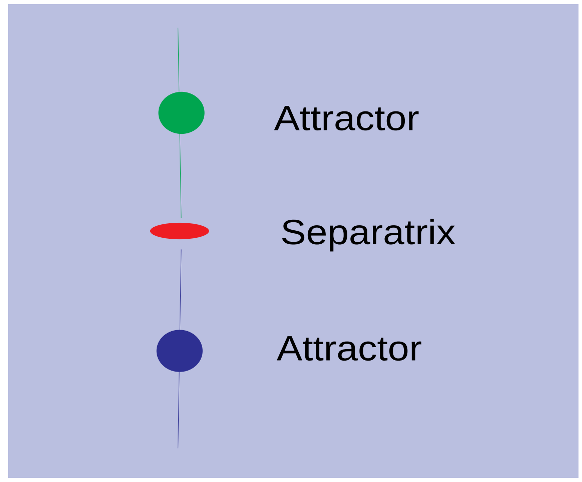


Figure 2. Cell F2: flows, two-dimensional. Here we see two point attractors, each in a two-dimensional basin. The two basins are divided by a one-dimensional separatrix. (This is figure 1.6.10 of DGB2, p. 51.)

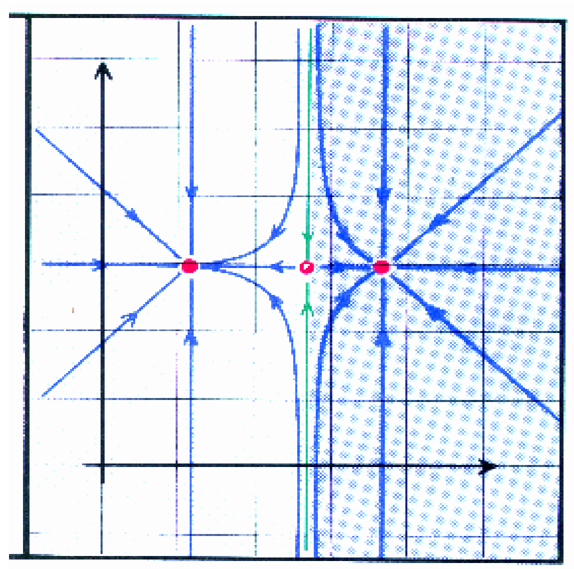


Figure 3. Cell F2: flows, two-dimensional. Here also we have two point attractors, each in a two-dimensional basin, and a one-dimensional separatrix. But in this example, the basins are thin, and mixed together. This is more realistic than Figure 2, in terms of the portraits which occur in real models. (This is figure 2.1.22 of DGB2, p. 64.)

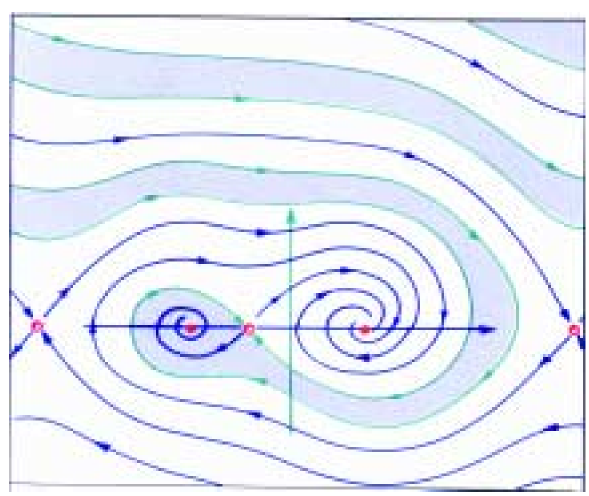


Figure 4. Cell F2: flows, two-dimensional, again. This time we have a single attractor, and it is periodic. Periodic attractors are models for oscillators. The basin is still two-dimensional. (This is figure 3.3.3 of DGB2, p. 104.)

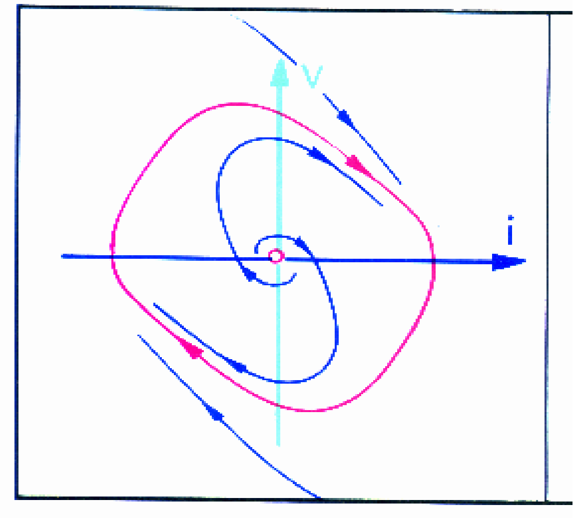


Figure 5. Cell F2: flows, two-dimensional, and:
Cell C1: cascades. one-dimensional.
In a context such as Figure 4, a point *P* is chosen on a periodic orbit, and a short line segment, or section, *S*, is drawn through *P* transverse to the periodic orbit. This drawing shows a blowup of the region near *P*. Points such as *x* and *y* on the section, *S*, are followed forward along their trajectories until they again pass through *S*, in the points *R(x)* and *R(y)*. This operation defines a map, *R*, of *S* to itself, called the *return map* by Poincaré. (This is figure 7.1.4 of DGB2, p. 237.)

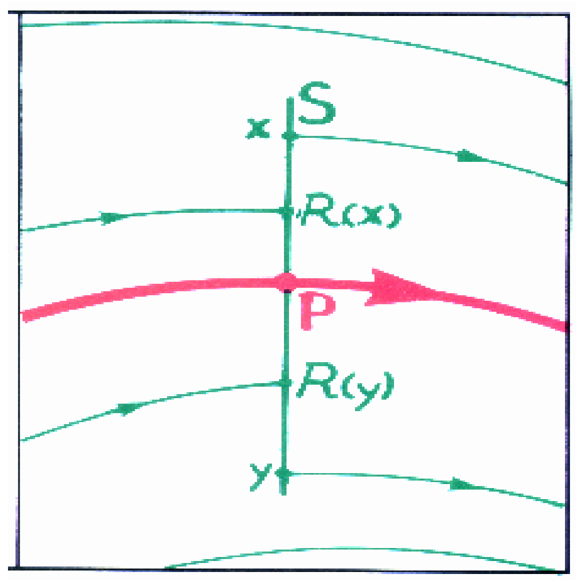


Figure 6. Cell F3: flows, three-dimensional, and:
Cell C2: cascades, two-dimensional.
This is the Poincaré section construction, as above, but one step to the right. We see three neighboring periodic orbits of a flow in a three-dimensional state space. The section is two-dimensional. (This is figure 4.3.13 of DGB2, p. 145.)

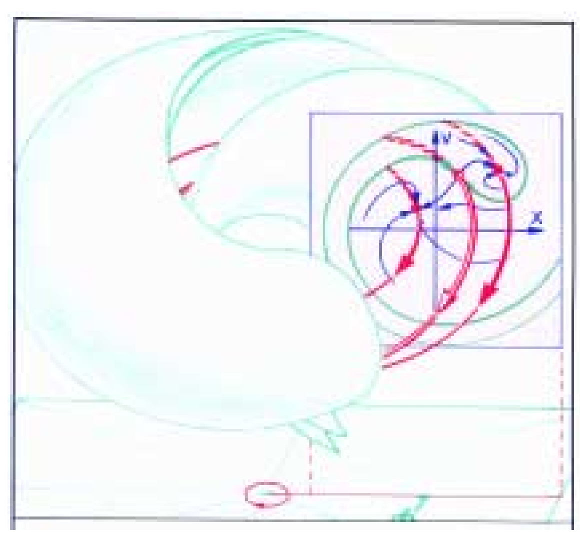


Figure 7. Cell F3: flows, three-dimensional. In this example, we see a chaotic attractor, the Rössler attractor. Its fractal dimension is bit more than two. (This is figure 8.4.9 of DGB2, p. 292.)

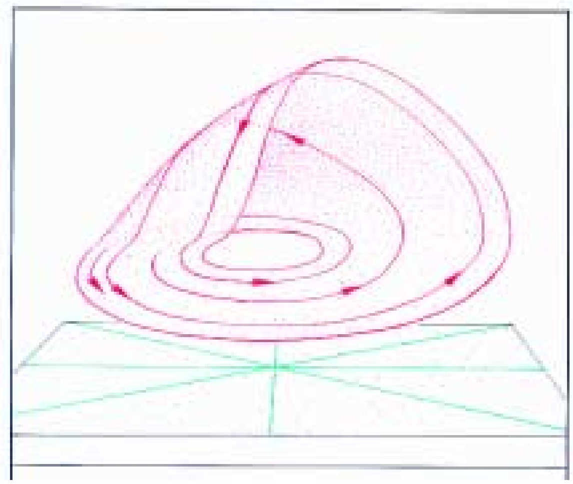


Figure 8. Cell F3: flows, three-dimensional. This is a schematic enlargement of a piece of the Rössler attractor, showing its fractal structure. (This is figure 9.4.4 of DGB2, p.319.)

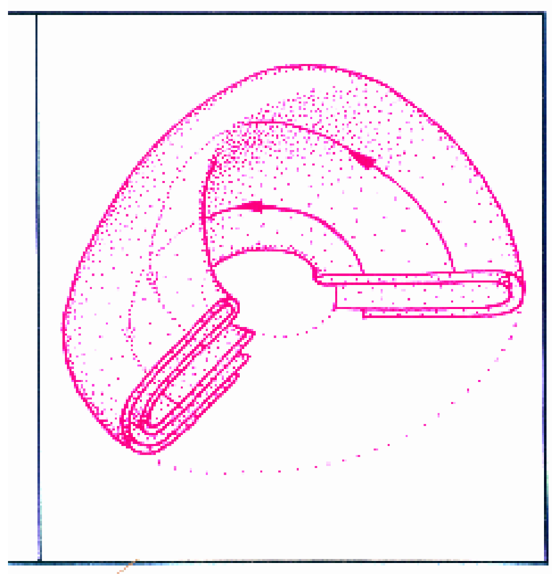


Figure 9. Cell F3: flows, three-dimensional, and: Cell C2: cascades, two-dimensional. Three is the lowest dimension, for flows, in which chaotic attractors appear. Similarly, two is the lowest dimension, for chaotic cascades. In addition, chaotic separatrices may occur. Here is an example of a fractal structure in a separatrix. (This is figure 14.1.9 from DGB2, p. 413.)

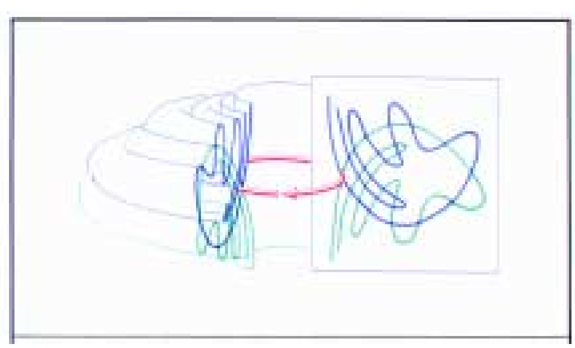


Figure 10. Cell S1: semicascades, one-dimensional. A quadratic, nonintervertible map, showing the graphical method of Koenigs-Lemaray for determining the trajectory of a point on the diagonal of the square. Such dynamical systems may have chaotic attractors. (This is figure 2-12 from JPX, p. 21.)

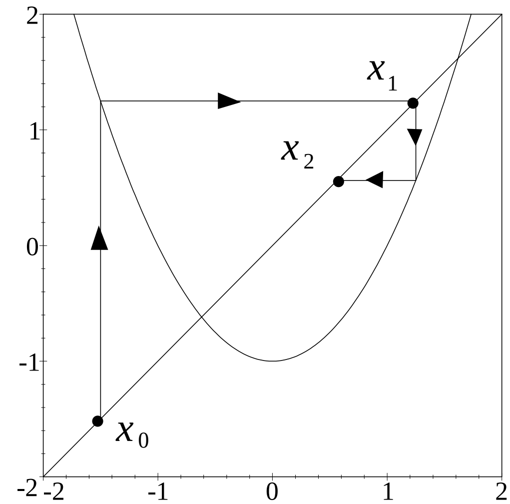


Figure 11. Cell S2: semicascades, two-dimensional. This context is on the frontier of chaos research today. Interaction between a chaotic attractor and a chaotic separatrix produce complex behavior. (This is figure7-24 from JPX, p. 144, drawn by Danielle Fournier-Prunaret.)

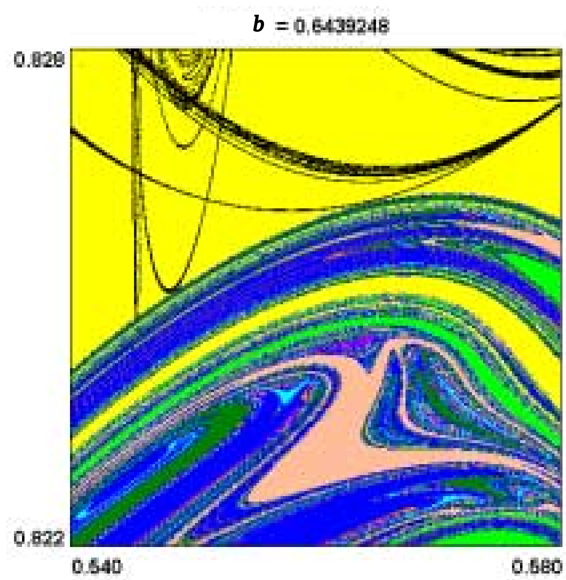
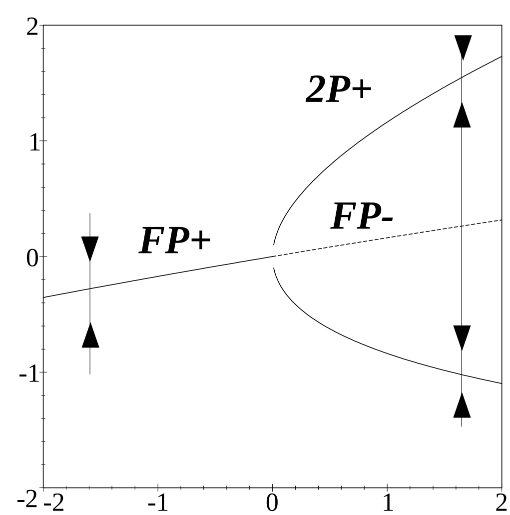


Figure 12. Cell S1: semicascades, one-dimensional. A dynamical scheme with one-dimensional state space (here shown vertical) and one-dimensional control space (horizontal). This is a bifurcation diagram showing the transformation of an attractive fixed point (*FP+*) into a repelling fixed point (*FP-*) and emitting a periodic attractor of period two *(2P+*). (This is figure 2-17 from JPX, p. 27.)

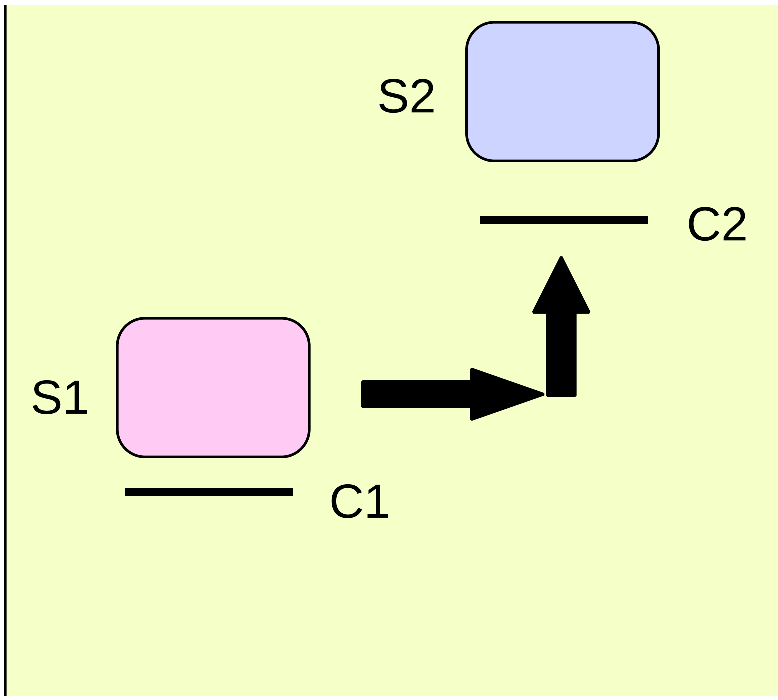


For more details of these concepts, see the references labelled DGB2 and JPX in the Bibliography.

4. Complex dynamical systems

The main idea of complex systems is the connection of dynamical schemes into a network. Connections are made from one scheme to another most simply by means of function from the state space of the first to the control space of the second. We may visualize this as a directed line segment, or arrow, from the response diagram (visualized as a rectangle, as in Figure 12) of the first to that of the second, as shown in Figure 13.

Figure 13. A serial link from the states of one scheme to the controls of another.



Many schemes may linked with connections such as this, according to a network, or directed graph. The result is a complex dynamical system. (For more details, consult CDS.)

5. Dynamical literacy and education

What we have seen here is barely the beginning of an elementary course in *dynamical literacy.* This means, according to Ervin Laszlo who has championed the idea, the understanding and use of dynamics concepts as a cognitive and linguistic strategy for perceiving, understanding, and discussing the complexities of the world in which we live. Such literacy may precede the study of the concepts in their mathematical form and context with great benefit. A holistic view of the world, consistent with the paradigm of general systems philosophy and theory, actual requires such a cognitive strategy, and thus, it should be a goal of our educational system. The chaos revolution has made this goal achievable, and software such as STELLA (from High Performance Systems, see www.hps-inc.com) may be used with great effect to these ends, as had been demonstrated in the high school programs of Diane Fisher (www.teleport.com/~sguthrie/ccstadus.html). among others.

6. Conclusion

We now return to the program of the unification of the sciences, with the basic concepts of the theories of chaos, bifurcation, and complex dynamical systems in hand. As the specialties of reductionistic science are replete with dynamical models which are dynamical schemes, we need only put them together to obtain holistic models. In fact, this has been an ongoing program in the frontiers of system dynamics: Jay Forrester and the Club of Rome come to mind. What is new now, after the chaos revolution, is the paradigm, the technical tools, and the large machines, which are required to deal with the inevitably chaotic behavior of these massive models.

Personally I am convinced that for general systems theory to have a role in the creation of our future, it must take a place in the world educational system, and demonstrate its power against the challenge of our World Problematique.

Bibliography

Abraham, Ralph H., 1991. CDS, *Complex Dynamics*, Aerial Press, Santa Cruz.

Abraham, Ralph H., and Christopher D. Shaw, 1992. DGB2, *Dynamics, The Geometry of Behavior, Second edition*, Addison-Wesley, Reading MA.

Abraham, Ralph H., Laura Gardini, and Christian Mira, 1997. JPX. *Chaos in Discrete Dynamical Systems*, Springer-Verlag, New York.

Laszlo, Ervin, 1991. *The Age of Bifurcation*, Gordon and Breach, Philadelphia.

Von Bertalanffy, Ludwig, 1972. The history and development of general systems theory, in: *The Relevance of General Systems Theory; papers presented to Ludwig von Bertalanffy on his seventieth birthday.* Edited by Ervin Laszlo, Braziller, New York.