Landscape Dynamics, Complex Dynamics, and Agent Based Models

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RUNNING HEAD: Landscape Dynamics.

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Abstract: *Evolutionary game models normally have a finite set of strategies. Recently, evolutionary games with continuous spaces of strategies have been introduced by Dan Friedman and Joel Yellin, under the name* landscape dynamics*. In one example extensively studied by them,* conspicuous consumption*, a simple (gradient) dynamical system is introduced on the strategy space. In this article, we translate a landscape dynamic system into a complex dynamical system, and then into an agent based model. The conspicuous consumption model of Friedman and Yellin is realized as an agent based model in NetLogo to illustrate the concepts of this new modeling strategy.*

Key Words: Evolutionary game theory, landscape dynamical systems, complex dynamical systems, agent based models, NetLogo.

INTRODUCTION

First, we recall the basic concepts of evolutionary game theory (EGT), landscape dynamical systems (LDS), complex dynamical systems (CDS), and agent based models (ABM). Along the way we will describe the process of discretization as required for a computational approach. In the next two sections we will describe the two-step translation, LDS to CDS and CDS to ABM, and the Veblen example.

Evolutionary Game Theory (EGT)

Evolutionary game models analyze strategic interaction over time. Equilibrium emerges, or fails to emerge, as players adjust their strategies in response to the payoffs they earn. Typically, a large number of players chose among a finite set of discrete strategies.

The instantaneous state of the system consists of the proportion of the population playing each of the pure strategies. This state is a density, a real-valued function on the finite set of strategies, and thus, a vector in a finite–dimensional space, the *state space*, A. The state evolves in continuous time, following some dynamical process that respects three principles. The process is (a) monotone: higher payoff strategies become more prevalent over time; (b) continuous (or inertial): only small changes in the state are possible in a very short time period; and (c) players do not try to influence the future play of others. We may view this as a dynamical system on the finite-dimensional state space.

In typical biological applications, the players are born, reproduce, and die. Each child has the same strategy as its parent. Meanwhile, players meet at random and play a fixed two-player game whose payoffs represent the players' fitness (number of offspring). An alternative narrative, more attuned to social science applications, is that the players learn and tend to switch to higher payoff strategies as they participate in some sort of ongoing interaction.

Landscape Dynamical Systems (LDS)

An LDS is a natural extension of EGT, in which players choose within a continuous strategy space, A. Typically, this is an open set in a Euclidean space, or an open set with its boundary. There are already a few papers that treat evolutionary games with continuous strategy spaces. Friedman and Yellin (1997, 2000) lay the groundwork for our approach.

The instantaneous state of the system is the density of players' choices in the strategy space, as above, or more commonly, the cumulative distribution (or integral) of that density. This is a real-valued function on the state space, A. While the instantaneous state in finite-choice EGT is a finite-dimensional vector, in the LDS context, it is a vector in an infinite-dimensional vector space, a function space.

A player's fitness (payoff) depends on her choice of strategy, as well as on the current state of the entire system. From the player's perspective, fitness looks like a landscape in which she seeks to go uphill. As play evolves over time, all players continuously adjust their strategies to increase their fitness, so the population distribution changes, and consequently the landscape morphs. A player's fitness changes as the direct result of her own actions, and also indirectly as the state evolves. The dynamics arise from this interplay between state and landscape.

The mathematical description of the basic landscape dynamical system is as follows. Let the unit interval $A = [0, 1]$ be the space of strategies, and let $\mathcal{F}$ be the space of all cumulative distribution functions on A, with the weak-star topology. That is, $F \in \mathcal{F}$ if F is a right-continuous nondecreasing function on A with range $[0, 1]$, and countable discontinuities. Recall that $F(x)$ represents the fraction of players whose choices are no larger than x. The space $\mathcal{F}$ is an infinite-dimensional simplex. More details may be found in Friedman and Abraham (2004).

The *fitness* for any player choosing strategy $x \in A$, when the current state is $F \in \mathcal{F}$, is denoted by $\phi(x, F)$. The application dictates a particular fitness function $\phi : A \times \mathcal{F} \to R$. The derived function $\phi^F : A \to R$; $x \mapsto \phi(x, F)$ with F held constant, is the instantaneous *fitness landscape* for that fitness function. It depends on F only, and we think of it as a landscape, or graph in $A \times R$.

The dependence of ϕ on F can take many forms. In some applications it is the expectation $\phi(x, F) = \int g(x, y)dF(y)$ of some two-player game payoff function $g(x, y)$. In some (possibly oversimplified) biological applications the dependence is only via the mean strategy $\mu_F = \int ydF(y)$. In fluid dynamics applications (and in the Veblen consumption application presented below) it depends only on the local value $F(x)$ at the chosen strategy x. In other economic applications the dependence on the current state F is quite nonlinear and arises from market-clearing prices.

Landscape dynamics are given as a vectorfield (defined almost everywhere) on the infinite-dimensional simplex. A general expression is

$$F_t(x, t) = \Psi(x, F, \phi) \tag{1}$$

where $F(x, t)$ is the cumulative distribution function over $x \in A$ for the state at time t, and F_t denotes the partial derivative of $F(x, t)$ with respect to t.

Consistent with the inertial principle of evolutionary games and with Darwin's dictum *Natura non facit saltum*, we assume that individual adjustment is continuous. That is, a discrete change in a player's strategy x takes a positive amount of time. And consistent with the monotone principle of evolutionary games, we assume that the direction of adjustment is given by the sign of the gradient, i.e., uphill in the fitness landscape. If also the adjustment speed is proportional to the gradient $\phi_x = \partial\phi/\partial x$, we have a *gradient adjustment system*. In this case dynamics obey the *master equation*,

$$F_t(x, t) = -\phi_x(x, F)F_x(x, t) \tag{2}$$

where F_x and F_t respectively denote the partial derivatives of $F(x,t)$ with respect to x and t. This nonlinear partial differential equation simply states that probability mass is conserved: the rate of change $F_t(x,t)$ in population mass to the left of any point x is equal to the (negative of the rightward) flux past that point. The flux is the product of the density $f = F_x$ and the velocity given by the gradient ϕ_x.

Discretization

For the digital simulation of an LDS, as always in computational mathematics, we discretize some of the continuous variables. In our case, the strategy space, A, is replaced by a discrete lattice, or regular grid of points in Euclidean space. The adaptive lattice system is identical to the adaptive landscape system, except that the continuous space of strategies, A, is replaced by a regular lattice of points, K in A. This seems, at first, identical to the basic EGT model above, which also has a finite set of strategies. However, here we may carry over the geometry of A to the finite subset, K. This will always be the case when we wish to simulate an adaptive landscape on a digital computer. These systems were introduced in Friedman and Yellin (1997) for simulation and analysis, under the name *discrete gradient dynamics.*

The category of adaptive lattice games is closely related to other modeling strategies for complex dynamical systems. For example, if the discrete action space is regarded as a physical substrate, the adaptive lattice system becomes a special type of agent based model, in which the global distribution of agents plays a dynamical role.

In our case, the lattice is a discretized form of the continuous space of actions. Thus, the choice of an action or strategy by a player or agent may be considered as a motion of the player over the lattice. This interpretation puts our discretized adaptive lattice model exactly in the context of ABM, permitting programming of our models in any ABM environment, for example, NetLogo.

The computer graphic features of ABM programming environments such as NetLogo greatly facilitate the visualization of the model as it runs. In the case of a one-dimensional or two-dimensional action space, NetLogo provides extensive graphics capabilities and is capable of animated representations of the LDS in play.

Complex Dynamical Systems (CDS)

A dynamical scheme is a dynamical system with control parameters. The dynamical schemes usually found in elementary texts are simple schemes. Their state spaces have low dimension, and they have just a few control parameters. They are suitable for modeling only the simplest natural systems, such as a pendulum, and are usually imaged as a bifurcation diagram.

More complex natural systems require model schemes made by combining several (or many) simple dynamical schemes into a network. A CDS is a network of dynamical schemes coupled by functions from the output of one node to the control parameters of another. See Abraham (1998).

One begins with a directed graph, that is, a diagram with blank boxes, *nodes*, connected by arrows, or*connections*. The nodes, corresponding to subsystems of the natural system, must be filled in with specific dynamical schemes. The connections must be specified by coupling functions, which enslave some controls (inputs) of a target scheme to the states (outputs) of a source scheme.

After being connected in this way, some of the controls of the node schemes are enslaved, and are thus no longer control parameters. Other node controls remain free. Thus, the fully connected complex scheme is still a dynamical scheme. The meaning of complex in this context refers to the means of construction of the model, as a system of subsystems.

Neural networks are complex dynamical schemes. So are most models in mathematical biology, ecology, atmospheric science, and so on. The evolving experience with massively complex schemes has led to *connectionism*, the idea that the connection graph is more important than the choice of models for the nodes.

Agent Based Models (ABM)

An ABM has intelligent agents moving in a geometrical space according to rules. We consider here an ABM that is a CDS with an auxiliary geometric space, called the *substrate*. Each node (dynamical scheme) of the CDS has a location in the substrate. The substrate coordinates of the location of a node are among its state variables. The coupling functions, as usual for a CDS, are functions from the state of one node into the control space of another.

In the NetLogo environment, the substrate is discretized in *patches*, the mobile agents are called *turtles*, and their are *observers* that are agents living in heaven, that is, outside the substrate. All three of these are agents, that is, they may contain dynamical systems, or other decision rule systems.

TRANSLATION

All three – LDS, CDS, ABM – are important modeling strategies for the social and behavioral sciences. In this paper we bridge the three strategies. We will translate first from LDS to CDS, and then to ABM.

From LDS to CDS

For the simulation of an evolutionary game, it is practical to reduce the number of players to a computationally practical size. Introducing dynamical systems on the common strategy space, we may ask that each player follow trajectories of one of these dynamical systems. Thus, dynamical systems replace the rules of an evolutionary game. We proceed with the translation of an LDS of the gradient adjustment type.

As we have a finite set of players, the cumulative distribution function, F, as a real-valued function on the continuous strategy space, A, will be piecewise constant, with jump discontinuities only at a finite number of points. We may now regard the players as the nodes of a complex dynamical system, as follows.

Consider a star-shaped network, with the players arranged in a circle about a single static node, the *observer*. A static node is simply a function. Each player is connected to the observer, and vice versa. There are no other connections between players. Each player is a dynamic node, that is, has a simple dynamical scheme. The strategy space, A, is the state space for each node. After discretizing time, the state of our system evolves dynamically by discrete timesteps.

After each timestep, each node reads out its strategy choice (part of its local state as a dynamical scheme) to the observer. The observer sums these up for all nodes, obtaining the cumulative distribution, F, or global state as an LDS. Then the observe, using the fixed payoff function, ϕ, and the current distribution, F, computes the fitness landscape function, ϕ^F, and its gradient, ϕ^F_x. This last, discretized as a lookup table of values, is then sent back from the observer to each node. The system is then ready for the next step, in which each player (dynamical node) advances uphill on the common landscape.

From CDS to ABM

Now that we have a moderate number of players following a common dynamical system on the strategy space, we may interpret everything from the ABM point of view. That is, the players are agents, the strategy space is their substrate, and the movements of the agents within the substrate is dictated by the given dynamical system. Now we have completed the two-step translation from LDS, through CDS, to ABM.

One further step of complexity allows us to translate any CDS (eg, a neural network) to an ABM. And that is, we allow each agent (player) to have her own dynamical scheme on the common ambient space. Coupling functions from the common ambient space to the controls of these individual control spaces complete the description of an almost-general complex dynamical system. The only special feature is that all nodes have the same state space, but this is a common situation. Each individual dynamical scheme (node) has its own bifurcation diagram.

Now we have succeeded in combining LDS, CDS, and ABM. To clarify the situation, an example from social economics, called conspicuous consumption by Rea (1834) and popularized by Veblen (1899), is presented below.

THE VEBLEN EXAMPLE

Veblen consumption illustrates several of possibilities inherent in landscape dynamics, and is interesting in its own right. Thorstein Veblen (1899) popularized the idea that some goods and services (think of Hummers or seldom-used second homes) are consumed largely to gain status. Such consumption has the desired effect only to the extent that it exceeds the conspicuous consumption of other people, i.e., its utility is rank-dependent.

Veblen as an LDS

Consider a single population of consumers with identical incomes. Each consumer chooses a fraction $x \in [0, 1]$ of income to allocate to ordinary consumption, and allocates the remaining fraction $1 - x$ to rank-dependent consumption. The state is the cumulative distribution function $F(x)$ of ordinary consumption. Assume standard direct utility, $c \ln x$, from ordinary consumption x, where the parameter $c \geq 0$ represents the relative importance of ordinary consumption. Suppose that rank-dependent utility arises from envy, i.e., I compare my rank-dependent consumption $1 - x$ to everyone else's, and am unhappy to the extent that it falls short. The shortfall is $\min\{0, y - x\}$ when your rank-dependent consumption is $1 - y$. After integrating the expected shortfall by parts, one verifies that overall expected utility is

$$\phi(x, F) = c \ln x - \int_0^x F(y)dy, \tag{3}$$

with gradient

$$\phi_x = c/x - F(x). \tag{4}$$

Dynamics are governed by the Master Equation (**??**). We insert the gradient (**??**) into (**??**) to obtain the partial differential equation

$$F_t = F_x[F - (c/x)]. \tag{5}$$

Friedman and Yellin (1997, 2000) have proved analytically that, under reasonable assumptions, this LDS has a single, global, point attractor, in which all consumers congregate at a single point in the strategy space. That is, (**??**) has a unique solution $F(x, t)$ starting from an arbitrary initial distribution, and as $t \to \infty$ the solution converges to the degenerate distribution at some point $\widetilde{x} \in [0, 1]$ whose value depends on the initial distribution and the parameter c.

Numerical explorations reported below suggest that even for moderately large weights c shock waves will arise from any local maximum of the initial density far enough above the point $\widetilde{x}$ where the gradient is zero.

Numerical Simulations

We now translate the LDS model conspicuous consumption into the world of computational mathematics, as an ABM. We will treat, in order, the density f, the distribution F, the payoff ϕ, and the gradient, ϕ_x. Then, we describe *Veblen*, our NetLogo model that implements the integration of an arbitrary initial distribution, to simulate the conspicuous consumption ABM. It will be convenient to collect here some basic equations from the preceding.

The density, f, represents a probability measure,

$$\int_0^1 f(y)dy = 1. \tag{6}$$

This function is nonnegative. The cumulative distribution is its integral,

$$F(x) = \int_0^x f(y)dy, \tag{7}$$

This function is nondecreasing, with clearly, $F(0) = 0$ and $F(1) = 1$. The payoff, as a function of x and F, is,

$$\phi(x, F) = c \ln x - \int_0^x F(y)dy. \tag{8}$$

Here c is a nonnegative constant. This function is nonpositive on the interval $(0, 1]$. Note the second integral. The gradient of ϕ is,

$$\phi_x = c/x - F(x). \tag{9}$$

Discretization

Let us suppose that the number of consumers is M, and that the strategy space, $A = [0, 1]$, is divided into N equal intervals. The width of these intervals is thus $\Delta x = 1/N$. Let m_i denote the number of consumers in the $i - th$ interval. Note

$$\sum_{i=0}^{N-1} m_i = M.$$

Formally, these intervals are closed on the left end, and open on the right, except for the first one, which is open on both ends, and the last one, which is closed on both ends. Informally, we may ignore this subtlety, and just imagine that there be no consumer on an endpoint.

Let f_i denote the average density of consumers in the $i - th$ interval, $[i\Delta x, (i+1)\Delta x)$, where $i = 0, ..., N - 1$. Then we have,

$$f_i = m_i/(M\Delta x) = (m_i/M)N \tag{10}$$

As a check, we numerically integrate f over $[0,1]$, obtaining,

$$\sum_{i=0}^{N-1} f_i \Delta x = \sum_{i=0}^{N-1} m_i/M = M/M = 1.$$

to confirm that f represents a probability measure, eqn. (**??**).

The cumulative distribution, F, is the integral of f according to eqn. (**??**), or, letting x_n denote the left endpoint of the $n-th$ interval,

$$F_n \equiv F(x_n) = \sum_{i=0}^{n-1} f_i \Delta x = \Delta x \sum_{i=0}^{n-1} f_i = (1/M) \sum_{i=0}^{n-1} m_i \tag{11}$$

and clearly, $F(x_0) = 0$ and $F(x_{N-1}) = 1$.

Fixing the distribution, F, give ϕ the value ϕ_i on the $i-th$ interval of the strategy space. Again, let x_i denote the left endpoint of the $i-th$ interval. Then we have, from eqn. (**??**), the average payoff on the $n-th$ patch is,

$$\phi_n = c \ln(x_n) - \sum_{i=0}^{n-1} F_i \Delta x = c \ln(x_n) - \Delta x \sum_{i=0}^{n-1} F_i = c \ln(x_n) - (1/N) \sum_{i=0}^{n-1} F_i \tag{12}$$

We display the payoff function in the simulation, but use only its gradient. Note that the payoff function defined thusly is piecewise constant. It is constant on each patch, so increasing the number, N, of patches on a row results in a better approximation of a smooth function. In the illustrations below, in which the graph of ϕ looks smooth, $N = 35$.

Using the notations above, throughout the $n-th$ interval of the strategy space, the average gradient is,

$$(\phi_x)_n = c/x_n - F_n, \tag{13}$$

from eqn. (**??**).

Our NetLogo Model

NetLogo displays an interface including control widgets, plots, monitors, a command center, and a two-dimensional graphics window called the *screen*. The screen comprises a rectangular grid of *patches*, which are squares of pixels. One may deploy mobile agents, *turtles*, in the screen, with floating point coordinates. We have used one row of patches to model our strategy space, and a number of turtles to model consumers.

Our NetLogo implementation closely follows the numerical equations above. We do not use the Master Equation for the evolution of F, but instead use a primitive integration in which each consumer changes her strategy according to the gradient rule, as follows. Each consumer has a numerical ID, an integer, and a position, a floating point number. For each discrete time step of size *stepsize*:

- Each consumer (in numerical order) moves $stepsize * (\phi_x)_n$ within the strategy space, where n is the index of the patch containing her current strategy, x.
- After all consumers have adjusted in this way, the density, f, distribution, F, payoff, ϕ, and gradient, ϕ_x, are recomputed.
- The time is incremented by *stepsize*.
- After each tenth step, the two plots are updated.

Given an initial distribution of consumers, upon clicking the "go" button, our program proceeds step-by-step, until stopped with another click on the "go" button.

The interface of Veblen is shown in Figure 1. On the upper left are a number of controls that allow the operator to approximate an arbitrary initial distribution of consumers, as follows.

1. There are five parallel rows, all of which are overlays of the strategy space. Using the pop-down menu labeled "puff-row", choose a row.
2. Choose a number of consumers to add to the initial distribution on the chosen row, using the slider labeled "population".
3. Choose a subinterval of the strategy space in which to randomly locate them, using the sliders labeled "center" and "width". Both are calibrated in percent of the unit interval.
4. Push the button labeled "setup".
5. Repeat 1-2-3, then push "puff" to add another square wave of consumers, as many times as you wish.

Mathematically, all consumers are on the same strategy space, the gray row. But the consumers are shown, as colored triangles, in five puff-rows on the screen. The two plots show the density and the landscape for the initial distribution. The color bar at the bottom of the screen shows where the slope is positive (magenta), zero (yellow), and negative (cyan).

On the lower left are controls for the step-by-step integration. Set "amp" (the parameter "c" in the model) and "stepsize" (proportion of slope for a consumer to move). Then push "step" for one step of integration, or "go" for a sequence of steps. These continue until "go" is pushed again. The current step number and time are shown in the monitors labeled "totalsteps" and "totaltime". A run may be continued by again pushing "go". (The command center is for NetLogo experts.)

The Hump Initial State

The hump is a heap, under an inverted parabola, as shown in the screen (upper right) of Figure 1. This is a piecewise constant approximation of the density $f(x, 0) = 6x(1 - x)$. After a long run it settles down to a static attractor, as shown in Figure 2. The transient behavior of the herd is interesting, as discussed above. A movie of this transient may be found at our website:

http://www.vismath.org/research/landscapedyn/models/veblen/hump01.gif

This movie, lasting only a few seconds, compresses an integration run that took about half an hour on our desktop computer. The number of consumers in this run was about 400. In future, we plan to accelerate these integrations by parallelizing the model with a cluster of computers. This will be particularly important when the number of agents is much larger, or if different initial distributions must be studied.

In case another model of landscape dynamics is to be studied numerically, then there may in general be several attractors with complicated basins, and it will be necessary to experiment with many initial distributions.

More details concerning landscape dynamics, the Veblen model, our NetLogo implementation applet, and the program in full, may be found at our website,

http://www.vismath.org/research/landscapedyn.

CONCLUSION

Our two goals here are present the analytic version of LDS, a new modeling strategy for the social and behavioral sciences, and the translation of an LDS into an ABM, or computational equivalent. The translation of an LDS into a computationally feasible ABM is a crucial part of the LDS technology. ABM working environments, such as NetLogo, make these simulations easy, with drag-and-drop tools such as graphics, plots, monitors, and so on. The Veblen example is primarily pedagogic, but models of greater scientific interest are sure to follow.

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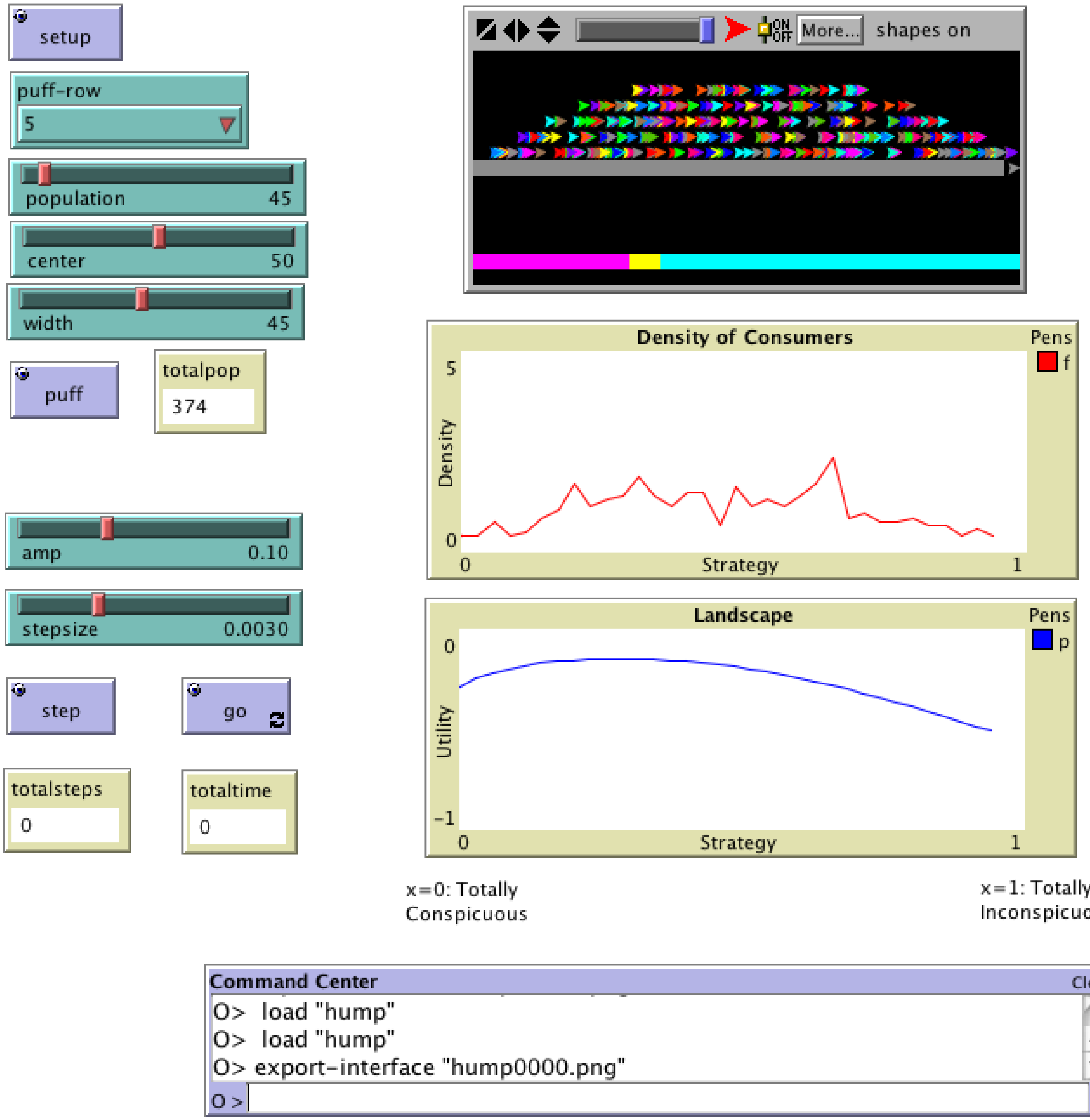

Figure 1: Interface of NetLogo model, showing the initial distribution.

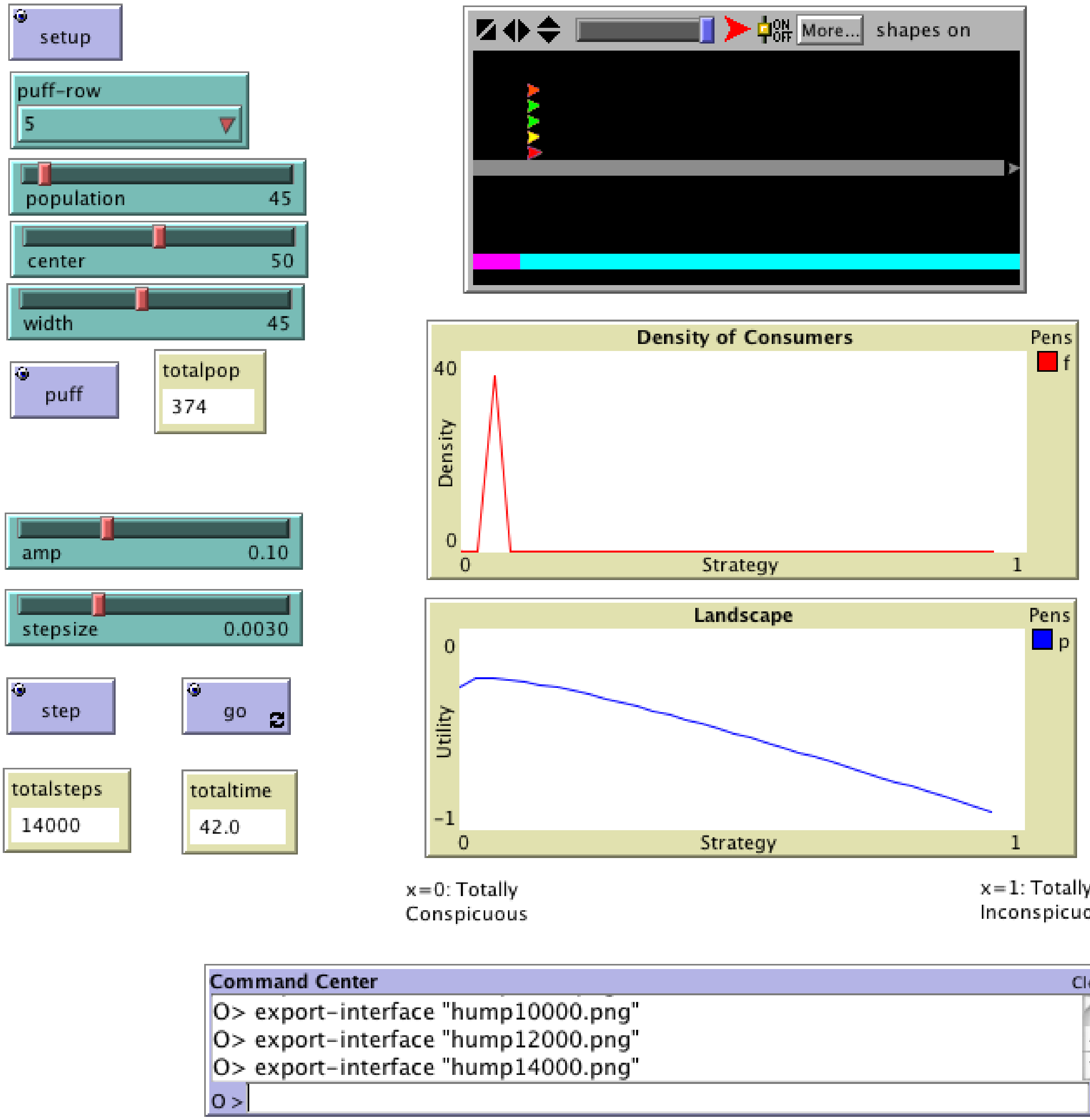

Figure 2: Interface of NetLogo model, showing the final distribution.